PHYS480/581 Cosmology
Distances in an Expanding Universe III

(Dated: September 26, 2022)

I. KEY DISTANCES THAT ARE OBSERVABLE

A. Angular diameter distance

Last time, we derived the angular diameter distance $d_{\rm A} \equiv l/\theta$, which allows one to relate the apparent angular size θ of an object on the sky to its physical size l. It is given by

$$d_{\rm A} = aS_k(\chi) = \begin{cases} \frac{a}{\sqrt{\Omega_K}H_0}\sinh\left(\sqrt{\Omega_K}H_0\chi\right) & \text{if } \Omega_K > 0, \\ a\chi & \text{if } \Omega_K = 0, \\ \frac{a}{\sqrt{|\Omega_K|}H_0}\sin\left(\sqrt{|\Omega_K|}H_0\chi\right) & \text{if } \Omega_K < 0. \end{cases} \tag{1}$$

where the comoving distance χ is given by

$$\chi(a) = \int_a^1 \frac{da'}{a'^2 H(a')} = \int_0^{z(a)} \frac{dz'}{H(z')}. \tag{2}$$

In the spatially flat case ($\Omega_K = 0$), the angular diameter distance is simply a rescaled version of the comoving distance

$$d_{\rm A}^{\rm flat}(z) = a\chi(z) = \frac{\chi(z)}{1+z}. \tag{3}$$

As we discussed in class, the angular diameter distance measures how far an object was from us at the time that the light we see today was *emitted* by that object.

B. Luminosity distance

Another very important way to infer distances in cosmology is to measure the flux from an object of known luminosity. In flat Euclidean space, the observed flux F a distance d from a source of known luminosity L is

$$F = \frac{L}{4\pi d^2}. \tag{4}$$

Here, L has units of energy per unit time, while F has units of energy per unit time per unit area. In analogy to the above, we define the *luminosity distance* $d_{\rm L}$ in an expanding universe as

$$d_{\rm L}^2 \equiv \frac{L}{4\pi F}, \tag{5}$$

where L is the absolute luminosity of the source and F is the observed flux. The challenge is then to relate the flux F to the absolute luminosity in an expanding Universe. Imagine the source emits dN_γ photons in a small time interval $dt_{\rm em}$ centered on time $t_{\rm em}$, each with energy $E_{\gamma,{\rm em}}$. The absolute luminosity of the source can then be written as

$$L = L(t_{\rm em}) = \frac{dN_\gamma}{dt_{\rm em}} E_{\gamma,{\rm em}}. \tag{6}$$

These photons will then propagate outward in a thin comoving shell surrounding the source. As they do so, the number of photons is conserved, but the total luminosity that any distant observer would infer will not equal L. To see this, consider an observer at time $t_{\rm obs} > t_{\rm em}$. This observer infers a total luminosity given by

$$L(t_{\rm obs}) = \frac{dN_\gamma}{dt_{\rm obs}} E_{\gamma,{\rm obs}}. \tag{7}$$

where $E_{\gamma,{\rm obs}}$ is the energy of the observed photons. Since photons travel on null paths, we have

$$dt_{\rm obs} = a(t_{\rm obs})d\chi = \frac{a(t_{\rm obs})}{a(t_{\rm em})} dt_{\rm em}. \tag{8}$$

Also, photons lose energy due to the expansion and we have

$$E_{\gamma,\mathrm{obs}} = E_{\gamma,\mathrm{em}} \frac{a(t_{\mathrm{em}})}{a(t_{\mathrm{obs}})}. \tag{9}$$

Putting all of this together, we have

$$L(t_{\mathrm{obs}}) = \left(\frac{a(t_{\mathrm{em}})}{a(t_{\mathrm{obs}})}\right)^2 \frac{dN_\gamma}{dt_{\mathrm{em}}} E_{\gamma,\mathrm{em}} = \left(\frac{a(t_{\mathrm{em}})}{a(t_{\mathrm{obs}})}\right)^2 L. \tag{10}$$

If the observation occurs today with $a(t_0) = 1$, this reduces to

$$L(t_0) = a(t_{\mathrm{em}})^2 L. \tag{11}$$

Since the comoving area of a sphere surrounding the source is $4\pi S_k^2(\chi(a(t_{\mathrm{em}})))$, the observed flux today is

$$F = \frac{L(t_0)}{4\pi S_k^2(\chi(a(t_{\mathrm{em}}))} = \frac{a(t_{\mathrm{em}})^2 L}{4\pi S_k^2(\chi(a(t_{\mathrm{em}})))}. \tag{12}$$

Substituting this in the definition of the luminosity distance, we obtain

$$d_{\mathrm{L}}(a) = \frac{S_k(\chi(a))}{a}. \tag{13}$$

In general, we thus have following relationship between luminosity and angular diameter distance

$$d_{\mathrm{L}}(a) = \frac{d_{\mathrm{A}}(a)}{a^2}, \tag{14}$$

or in terms of redshift

$$d_{\mathrm{L}}(z) = (1+z)^2 d_{\mathrm{A}}(z). \tag{15}$$

Astronomers like to use *magnitudes* to describe flux and luminosity. The *absolute* luminosity is defined as the log of the flux that would be measured if an object was 10 pc away.

$$M \equiv -2.5 \log_{10}\left(F(10\,\mathrm{pc})\right) + C, \tag{16}$$

where the leading factor of -2.5 is a convention and the constant C is used to set the reference point of the magnitude system. Conversely, the *apparent* magnitude is the log of the measured flux

$$m \equiv -2.5 \log_{10}(F) + C. \tag{17}$$

Going back to the definition of the luminosity distance, we can write

$$\left(\frac{d_{\mathrm{L}}}{10\,\mathrm{pc}}\right)^2 = \frac{L}{4\pi(10\,\mathrm{pc})^2 F}, \tag{18}$$

and note that $F(10\,\mathrm{pc}) = L/(4\pi(10\,\mathrm{pc})^2)$. Taking a $\log_{10}$ on both sides and multiplying by -2.5.

$$\begin{aligned} -5\log_{10}\left(\frac{d_{\mathrm{L}}}{10\,\mathrm{pc}}\right) &= -2.5\log_{10}\left(\frac{L}{4\pi(10\,\mathrm{pc})^2}\right) + 2.5\log_{10}(F) \\ -\mu &= M - m \\ \mu &= m - M, \end{aligned} \tag{19}$$

where μ is the *distance modulus*, which is defined as

$$\mu \equiv 5\log_{10}\left(\frac{d_{\mathrm{L}}}{10\,\mathrm{pc}}\right) = 5\log_{10}\left(\frac{d_{\mathrm{L}}}{\mathrm{Mpc}}\right) + 25. \tag{20}$$

Thus, if you happen to know the absolute luminosity M of a given type of objects, you can use observations of their apparent luminosities to measure the luminosity distances to these objects. Then by comparing these measured distances to their theoretical predictions (see Eq. (13)) above, one can infer useful information about cosmological parameters (e.g. H_0, Ω_{m}, etc.).