PHYS480/581 Cosmology
Distances in an Expanding Universe I

(Dated: September 19, 2022)

I. HUBBLE'S LAW

We have seen that the scale factor $a(t)$ links comoving distances to physical distances

$$r_{\rm phys}(t) = a(t) r_{\rm com}. \tag{1}$$

The rate of change of the physical distance is then

$$v_{\rm phys} \equiv \dot{r}_{\rm phys} = \dot{a} r_{\rm com} = \frac{\dot{a}}{a} r_{\rm phys} = H r_{\rm phys}, \tag{2}$$

since $r_{\rm com}$ is a constant comoving distance. This is called *Hubble's law*, and dictates that an object at a physical distance d away from us will be moving away from us with speed $v = Hd$. Thus, in an expanding universe, objects at a larger distance recede faster from us than closer objects. Today, objects situated at distances greater than $r_{\rm phys} = 1/H_0$ (or c/H_0 if you restore the factor of speed of light) appear to be moving away from us faster than the speed of light. This does not contradict special relativity since this is only an apparent relative velocity, and no inertial observer is seeing objects moving faster than the speed of light. However, this distance is so important that we give is a special name: the *Hubble radius* $r_H \equiv 1/H_0$. This distance tells us about the typical size of the Universe today.

II. REDSHIFT

Light propagating in an expanding Universe has its wavelength stretched by the expansion. We characterize this via the redshift z, which is defined as

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} \tag{3}$$

where $\lambda_{\rm obs}$ is the observed wavelength and $\lambda_{\rm em}$ is wavelength at emission. The wavelength is a physical distance (between two wave crests, say) and thus scales as any physical distance, that is, $\lambda \propto a$. We thus have

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})} \tag{4}$$

Typically, the observation time is simply $t_{\rm obs} = t_0$, and this reduces to $\lambda_{\rm obs}/\lambda_{\rm em} = 1/a(t_{\rm em})$, since $a(t_0) = 1$. This means that the redshift is simply related to the scale factor by

$$z = \frac{1}{a} - 1 \tag{5}$$

or in a more compact form

$$a = \frac{1}{1+z}. \tag{6}$$

Since $a(t_0) = 1$ today, this means that $z = 0$ today, and higher redshifts denote earlier times (with $z = \infty$ denoting the Big Bang). Just like a, z is often used as a time-variable, remembering that redshifts run backward. For instance, astronomers frequently use redshift to denote how old an object they are looking at is. We also often write the Friedmann equation in terms of redshift

$$H^2(z) = H_0^2 \left(\Omega_{\rm rad}(1+z)^4 + \Omega_{\rm m}(1+z)^3 + \Omega_K (1+z)^2 + \Omega_\Lambda \right). \tag{7}$$

III. COMOVING HORIZON

The comoving distance that a photon could have travelled from time $t = 0$ to any other time t (or scale factor a) is given by

$$\eta \equiv \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{a'^2 H(a')}, \tag{8}$$

where we have used the fact that photons always travel on null paths ($ds^2 = 0$). This distance is very important since no information could have propagated further on the comoving grid. Regions separated by distance greater than η cannot be in causal contact. This is why η is often referred to as the *comoving horizon*. Since η is a monotonically increasing variable, it is a useful time variable usually referred to as the *conformal time*.