PHYS480/581 Cosmology
The Fluid Equation

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I. CONSERVATION OF ENERGY

Last time, we derived the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2}, \tag{1}$$

where k is a constant with units of $[\text{length}]^{-2}$, which is related to the spatial geometry of the Universe. More on this constant later. Remember that the specific combination $\dot{a}/a$ appearing in the Friedmann equation is referred to as the *Hubble rate*

$$H \equiv \frac{\dot{a}}{a}. \tag{2}$$

This quantity characterizes the rate of expansion (or contraction) of the Universe. To solve the Friedmann equation, we will need to know the how $\rho(t)$ changes with time, that is, we need an evolution equation for $\rho(t)$ to close this system of equations. To derive such an equation, we start from the first law of thermodynamics, which is simply a statement about energy conservation.

$$dE + pdV = TdS, \tag{3}$$

where E is the total energy, p is the pressure, V is the physical volume, T is the temperature, and S is the entropy. Assuming an adiabatic reversible evolution, which is actually an excellent approximation for most important processes going on in cosmology, we set $dS = 0$ here. Now consider a fluid with energy density $\rho(t)$. This means $E = \rho(t)V(t)$. Now, we can write the physical volume V as

$$V(t) = a^3(t)V_{\text{com}}, \tag{4}$$

where V_{com} is a fixed comoving volume. In a small time interval dt, the change in energy dE is then,

$$dE = (\dot{\rho}a^3 + 3\rho a^2\dot{a})V_{\text{com}}dt \tag{5}$$

where we have used

$$dV = d(a^3V_{\text{com}}) = 3a^2\dot{a}V_{\text{com}}dt. \tag{6}$$

The first law of thermodynamics then takes the form

$$(\dot{\rho}a^3 + 3\rho a^2\dot{a})V_{\text{com}}dt + 3pa^2\dot{a}V_{\text{com}}dt = 0. \tag{7}$$

Dividing everything by $a^3V_{\text{com}}dt$, we get

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{8}$$

which is the desired evolution equation for $\rho(t)$. Note that this equation involves yet another quantity, the pressure p. To close the system of equation, we thus need an *equation of state* w relating the pressure to the energy density of whichever fluid we are considering,

$$w \equiv \frac{p}{\rho}. \tag{9}$$

While the equation of state w could technically have some time dependence, we will focus for now on components with constant equation of state. Some of these are listed in Table I. In this case, one can derive a simple expression for how the energy density ρ depends on the scale factor a of the Universe.

TABLE I. Equation of state for different possible components of the Universe.

Component type	$\boldsymbol{w}$
kinetic energy	1
radiation	1/3
non-relativistic matter	0
curvature	-1/3
dark energy	-1