PHYS480/581 Cosmology
The Friedmann Equation

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I. THE FRIEDMANN EQUATION

Last time, we argue that the cosmological principle leads to the Friedmann-Lemaître-Robertson-Walker metric, which in its spatially flat version (and written in cartesian coordinates) takes the form

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1}$$

where $a(t)$ is the scale factor. We would now like to compute the evolution of the scale factor $a(t)$. This technically requires General Relativity, which fundamentally relates the energy content of the Universe to its geometry and evolution. However, it turns out that a calculation based on Newtonian mechanics leads to essentially the same result, so we will follow that route instead.

A key element of Newtonian gravity is that the gravitational field (or acceleration) $\mathbf{g}$ obeys Gauss's law: If I integrate $\mathbf{g}$ over a closed surface S, the gravitational "flux" going through the surface depends only on the mass enclosed within this surface $M_{\rm enc}$, or mathematically

$$\oint_S \mathbf{g}\cdot d\mathbf{a} = -4\pi G M_{\rm enc}. \tag{2}$$

This is, of course, very similar to Gauss's law for the electric field, and is ultimately the consequence of both fields obeying a $1/r^2$ law. Note that the result above doesn't depend on the details of the surface S, as long at encloses the same mass $M_{\rm enc}$. For example, the gravitational field (or acceleration) a distance r away from a point mass M located at the origin is

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}, \tag{3}$$

and using a sphere of radius R as the surface S ($d\mathbf{a} = R^2 d\Omega\hat{\mathbf{r}}$, with $d\Omega = \sin\theta d\theta d\phi$), we obtain

$$\begin{aligned}\oint_S \mathbf{g}\cdot d\mathbf{a} &= -\int \frac{GM}{R^2}\hat{\mathbf{r}}\cdot R^2 d\Omega\hat{\mathbf{r}}\\ &= -GM\int d\Omega\\ &= -4\pi GM,\end{aligned} \tag{4}$$

which verifies the above as M is the only mass enclosed in this case. Since any finite mass distribution can be built by adding together point masses, the above can be generalized to arbitrary mass distributions. In particular, consider a universe filled with a uniform matter density $\rho(t)$. Pick an arbitrary origin in this universe (since it is assumed to be homogeneous, all choices of origin must be equivalent, and our final answer will not depend on this choice), and consider a test mass m a distance r away from that origin. Since the gravitational field obeys Gauss's law, only the mass contained within the sphere of radius r surrounding the origin can exert a force on the test mass m. This mass is simply

$$M = \int \rho(t) d^3r = \frac{4\pi r^3\rho(t)}{3}, \tag{5}$$

since the matter density $\rho(t)$ is uniform in space.

To get the evolution of the scale factor $a(t)$, we will examine the evolution of the energy of the test particle m. This particle has both gravitational potential energy and kinetic energy. Remembering that the gravitational potential Φ is related to the gravitational field by $\mathbf{g} = -\boldsymbol{\nabla}\Phi$, the gravitational potential at the location of the mass m is simply

$$\Phi = -\frac{GM}{r} = -\frac{4\pi G r^2\rho(t)}{3}. \tag{6}$$

Since the gravitational potential is the potential energy per unit mass, the potential energy V of the particle m is then simply

$$V = -\frac{4\pi G r^2 \rho(t) m}{3}. \tag{7}$$

Meanwhile, the kinetic energy of the particle m is

$$T = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2. \tag{8}$$

Then, the total energy U of this particle is then

$$U = T + V = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 - \frac{4\pi G r^2 \rho(t) m}{3}. \tag{9}$$

Here, r is a physical distance between the origin and the mass m. We now introduce comoving coordinates $\mathbf{r}_{\rm com}$

$$\mathbf{r} = a(t)\mathbf{r}_{\rm com}, \tag{10}$$

such that

$$\frac{dr}{dt} \equiv \dot{r} = \dot{a} r_{\rm com}. \tag{11}$$

The total energy is then

$$U = \frac{1}{2} m \dot{a}^2 r_{\rm com}^2 - \frac{4\pi G a^2 r_{\rm com}^2 \rho(t) m}{3}. \tag{12}$$

Dividing both sides by $a^2 r_{\rm com}^2 m/2$, we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2}, \tag{13}$$

where we have defined $k = -\frac{2U}{r_{\rm com}^2 m}$. Here, k is a constant with units of $[\text{length}]^{-2}$ whose physical meaning is a little mysterious at this point. As we will soon see, this constant is related to the spatial geometry of the Universe. It turns out that setting $k = 0$ correspond to having the flat FLRW metric given in Eq. (1) above.

Equation (13) is called the *Friedmann equation*. It is one of the most important equations in all of cosmology. It relates the energy density of the Universe to the behavior of the scale factor, such that universes dominated by different kind of energy (matter, radiation, dark energy) will behave differently.

The specific combination $\dot{a}/a$ appearing in the Friedmann equation is referred to as the *Hubble rate*

$$H \equiv \frac{\dot{a}}{a}. \tag{14}$$

This quantity characterizes the rate of expansion (or contraction) of the Universe. Why is H more important than $\dot{a}$ you may ask? It is because H is independent of how $a(t)$ is normalized, that is, it doesn't depend on having $a(t_0) = 1$ today. It is a true physical rate that we can measure (and we do!). It's value today, $H(t_0) \equiv H_0$, is usually referred to as the Hubble constant. It is customary to quote this rate in units of km/s/Mpc.

To solve the Friedmann equation, we generally need to know the how $\rho(t)$ changes with time, that is, we need an evolution equation for $\rho(t)$ to close this system of equations. We will discuss this equation in the next lecture.