PHYS480/581 Cosmology
Thermodynamics for Cosmologists III

(Dated: October 12, 2022)

I. NUMBER DENSITY, ENERGY DENSITY, AND PRESSURE

A reminder that the key expression for number density n, energy density ρ, and Pressure P are given by

$$n(t) = g \int \frac{d^3p}{(2\pi)^3} f(p,t), \tag{1}$$

$$\rho(t) = g \int \frac{d^3p}{(2\pi)^3} f(p,t) E(p), \tag{2}$$

$$P(t) = g \int \frac{d^3p}{(2\pi)^3} f(p,t) \frac{p^2}{3E(p)}, \tag{3}$$

respectively. Here, $f(p,t)$ is the particle distribution function, and g is the number of internal degrees of freedom.

If a particle species is in kinetic equilibrium (i.e. particles are able to efficiently exchange energy and momentum), then the particle distribution function takes either a *Fermi-Dirac* or *Bose-Einstein* form

$$f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}, \tag{4}$$

where the $+$ sign is for fermions (half-integer spin) and the $-$ sign for bosons (integer spin). Here, μ is the *chemical potential* and T is the temperature.

II. NON-RELATIVISTIC LIMIT

A. Number density

Let's consider the number density in the non-relativistic limit for a particle species of mass m. Starting with the relativistic expression for the energy $E = \sqrt{p^2 + m^2}$, we can Taylor expand to get

$$E \approx m + \frac{p^2}{2m} + \ldots \tag{5}$$

in the nonrelativistic limit $m \gg p$. Using this expression for E, the number density for non-relativistic particles ($m \gg T$) in thermal equilibrium is given by

$$\begin{aligned} n_{\rm NR} &\approx g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{(m+p^2/2m-\mu)/T} \pm 1} \\ &= \frac{g}{2\pi^2} e^{-(m-\mu)/T} \int_0^\infty dp\, p^2 e^{-p^2/(2mT)} \\ &= \frac{g}{2\pi^2} e^{-(m-\mu)/T} (2mT)^{3/2} \int_0^\infty dx\, x^2 e^{-x^2}. \end{aligned} \tag{6}$$

We can now use the following result

$$\int_0^\infty dx\, x^n e^{-x^2} = \frac{1}{2} \Gamma\left(\frac{1}{2}(n+1)\right), \tag{7}$$

where $\Gamma(z)$ is the gamma function. Also note that $\Gamma(3/2) = \sqrt{\pi}/2$. Putting everything together, we get

$$n_{\rm NR} = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}. \tag{8}$$

B. Energy density

Non-relativistic particles are dominated by their rest-mass energy $E \approx m$. Using this leading approximation, the energy density for a species of non-relativistic particles is

$$\rho_{\rm NR} \approx g \int \frac{d^3p}{(2\pi)^3} f(p,t) m = m \left(g \int \frac{d^3p}{(2\pi)^3} f(p,t) \right) = mn. \tag{9}$$

That is, for non-relativistic particles, the leading contribution to the energy density is simply their mass times their number density. In our Universe, this result is very relevant to the baryonic and dark matter densities. If we were to keep the kinetic energy contribution to the energy (second term in Eq. (5)), we would then have

$$\begin{aligned} \rho_{\rm NR} &\approx g \int \frac{d^3p}{(2\pi)^3} \frac{m + p^2/2m}{e^{(m+p^2/2m-\mu)/T} \pm 1} \\ &= mn + \frac{g}{2\pi^2} \frac{e^{-(m-\mu)/T}}{2m} \int_0^\infty dp\, p^4 e^{-p^2/(2mT)} \\ &= mn + \frac{g}{2\pi^2} \frac{e^{-(m-\mu)/T}}{2m} (2mT)^{5/2} \int_0^\infty dx\, x^4 e^{-x^2} \\ &= mn + \left[g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-(m-\mu)/T} \right] \frac{4T}{\sqrt{\pi}} \frac{1}{2} \Gamma(5/2) \\ &= mn + \frac{3}{2} nT, \end{aligned} \tag{10}$$

where we have used $\Gamma(5/2) = 3\sqrt{\pi}/4$. We thus retrieve the standard result that each particle has a typical kinetic energy of $(3/2)T$ (remember that we have set $k_{\rm B} = 1$ here). Since non-relativistic particles have $m \gg T$ the second term is generally extremely subdominant compared the rest-mass contribution.

C. Pressure

As we have already discussed, the pressure of non-relativistic particles is generally vanishingly small in a cosmological context. We can now quantify this statement more thoroughly. The leading order contribution to the pressure is given by

$$\begin{aligned} P_{\rm NR} &\approx g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{(m+p^2/2m-\mu)/T} \pm 1} \frac{p^2}{3m} \\ &= \frac{g}{2\pi^2} \frac{e^{-(m-\mu)/T}}{3m} \int_0^\infty dp\, p^4 e^{-p^2/(2mT)} \\ &= \frac{2}{3} \left[\frac{g}{2\pi^2} \frac{e^{-(m-\mu)/T}}{2m} \int_0^\infty dp\, p^4 e^{-p^2/(2mT)} \right] \\ &= nT, \end{aligned} \tag{11}$$

which is simply the familiar $PV = Nk_{\rm B}T$ ideal gas law. For $m \gg T$, this pressure is indeed much smaller than the energy density from the rest mass of the particles. The equation of state for non-relativistic particle is then, at leading order,

$$w = \frac{P_{\rm NR}}{\rho_{\rm NR}} \approx \frac{T}{m} \ll 1, \tag{12}$$

indeed indicating that $w \approx 0$ for non-relativistic matter is an excellent approximation.