PHYS480/581 Cosmology
Thermodynamics for Cosmologists II

(Dated: October 6, 2022)

I. NUMBER DENSITY, ENERGY DENSITY, AND PRESSURE

Last time, we introduced the number density

$$n(t) = g \int \frac{d^3p}{(2\pi)^3} f(p,t), \tag{1}$$

which is the number of particle of a given species per unit volume. Here, $f(p,t)$ is the particle distribution function, and g is the number of internal degrees of freedom. Remember that we have set $\hbar = 1$ here. We also introduced the energy density

$$\rho(t) = g \int \frac{d^3p}{(2\pi)^3} f(p,t) E(p). \tag{2}$$

where $E(p) = \sqrt{p^2 + m^2}$ is the energy. In the above, we have assumed that the particles are essentially free, that is, that we can neglect the interaction energies between the particles. This is usually a very good approximation in cosmology. Meanwhile, the pressure P was given by

$$P(t) = g \int \frac{d^3p}{(2\pi)^3} f(p,t) \frac{p^2}{3E(p)}. \tag{3}$$

We also discussed that if a particle species is in kinetic equilibrium (i.e. particles are able to efficiently exchange energy and momentum), then the particle distribution function takes either a *Fermi-Dirac* or *Bose-Einstein* form

$$f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}, \tag{4}$$

where the $+$ sign is for fermions (half-integer spin) and the $-$ sign for bosons (integer spin). Here, μ is the *chemical potential* and T is the temperature. Here, we have set the Boltzmann constant $k_{\rm B} = 1$, meaning that we are measuring temperature in units of energy (eV, say). In cosmology, if a species has a nonzero chemical potential, it means that the number of particles and of the corresponding anti-particles are different. For a particle species X and its anti-particle $\bar{X}$, we generally have

$$\mu_X = -\mu_{\bar{X}}. \tag{5}$$

II. RELATIVISTIC LIMIT

A. Number density

Let us first consider the relativistic limit $p \gg m$, such that $E \simeq p$. Here, we set the chemical potential to zero. The number density is then given by

$$\begin{aligned} n &= g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{p/T} \pm 1} \\ &= \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2}{e^{p/T} \pm 1} \\ &= \frac{gT^3}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^x \pm 1}, \end{aligned} \tag{6}$$

where we have changed the variable to $x = p/T$. We can now use the known result that

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = \zeta(n+1)\Gamma(n+1), \tag{7}$$

where $\zeta(z)$ is the Riemann zeta function and $\Gamma(n+1) = n!$ (for integer n) is the gamma function. For bosons, we immediately get the result

$$n_{\rm Bosons} = \frac{g\zeta(3)T^3}{\pi^2}. \tag{8}$$

For fermions, we can also use the above result once we notice that

$$\frac{1}{e^x+1} = \frac{1}{e^x-1} - \frac{2}{e^{2x}-1}, \tag{9}$$

which allows us to write

$$\begin{aligned}\int_0^\infty dx \frac{x^2}{e^x+1} &= \int_0^\infty dx \left[\frac{x^2}{e^x-1} - \frac{2x^2}{e^{2x}-1}\right] \\ &= 2\zeta(3) - 2\int_0^\infty \frac{dy}{2}\frac{(y/2)^2}{e^y-1} \\ &= 2\zeta(3)\left(1-\frac{1}{4}\right) \\ &= 2\zeta(3)\frac{3}{4}.\end{aligned} \tag{10}$$

For fermions, we thus get

$$n_{\rm Fermions} = \frac{3}{4}\frac{g\zeta(3)T^3}{\pi^2}. \tag{11}$$

We thus obtain the general behavior that $n \propto T^3$ for a species in thermal equilibrium. For example, the number density of CMB photons today ($g = 2$ for the two polarization, $T_0 = 2.725\text{K} = 2.348 \times 10^{-4}$ eV) is given by

$$n_\gamma(t_0) = \frac{2\zeta(3)T_0^3}{\pi^2} \simeq 410\,\text{photons/cm}^3, \tag{12}$$

where we have used $\hbar c = 1.97 \times 10^{-5}$ eV cm.

B. Energy density

Using $E(p) = p$, the energy density takes the form

$$\begin{aligned}\rho &= g\int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T}\pm 1} \\ &= \frac{g}{2\pi^2}\int_0^\infty dp \frac{p^3}{e^{p/T}\pm 1} \\ &= \frac{gT^4}{2\pi^2}\int_0^\infty dx \frac{x^3}{e^x \pm 1},\end{aligned} \tag{13}$$

where we have defined $x = p/T$. For bosons, we can directly used the result from Eq. (7) to obtain

$$\rho_{\rm Bosons} = 3!\frac{gT^4}{2\pi^2}\zeta(4) = 3\frac{gT^4}{\pi^2}\frac{\pi^4}{90} = g\frac{\pi^2}{30}T^4, \tag{14}$$

where we have used the fact that $\zeta(4) = \pi^4/90$. For fermions, we use the same trick as in Eq. (9), which allows us to write

$$\begin{aligned}\int_0^\infty dx \frac{x^3}{e^x+1} &= \int_0^\infty dx \left[\frac{x^3}{e^x-1} - \frac{2x^3}{e^{2x}-1}\right] \\ &= 3!\zeta(4) - 2\int_0^\infty \frac{dy}{2}\frac{(y/2)^3}{e^y-1} \\ &= 6\frac{\pi^4}{90} - \frac{1}{8}6\frac{\pi^4}{90} \\ &= 2\frac{7}{8}\frac{\pi^4}{30}.\end{aligned} \tag{15}$$

Thus, for fermions the energy density is

$$\rho_{\rm Fermions} = g\frac{7}{8}\frac{\pi^2}{30}T^4. \tag{16}$$

We this obtain the general behavior that $\rho \propto T^4$ for a species in thermal equilibrium. At the same temperature, a fermionic species has an energy density that is suppressed by a factor of 7/8 compared to a similar bosonic gas.

For example, the energy density in CMB photons today is

$$\rho_\gamma(t_0) = 2\frac{\pi^2}{30}T_0^4 = \frac{\pi^2}{15}T_0^4 = 2.0 \times 10^{-15}\,{\rm eV}^4. \tag{17}$$

Dividing this by the critical density of the Universe today $\rho_{\rm c} = 3H_0^2/(8\pi G) = 8.098h^2 \times 10^{-11}\,{\rm eV}^4$, we get

$$\Omega_\gamma = \frac{\rho_\gamma(t_0)}{\rho_{\rm c}} = 2.47 \times 10^{-5}h^{-2}, \tag{18}$$

which is the number we quoted before. Here, h is the reduced Hubble rate $h = H_0/(100\,{\rm km/s/Mpc})$.

C. Pressure

Using $E(p) = p$, the pressure takes the form

$$P = g\int \frac{d^3p}{(2\pi)^3}f(p)\frac{p}{3} = \frac{1}{3}g\int \frac{d^3p}{(2\pi)^3}f(p)p = \frac{\rho}{3}, \tag{19}$$

that is, we just retrieve the standard equation of state for relativistic particles $w = P/\rho = 1/3$.