PHYS480/581 Cosmology
Matter-radiation Equality and the Epoch of Recombination

(Dated: November 2, 2022)

I. THE UNIVERSE AFTER NUCLEOSYNTHESIS

Once nucleosynthesis is done, about 15 minutes after the Big Bang, the energy density of the Universe is still dominated by photons and relativistic neutrinos, i.e. the radiation of the Universe. The baryonic mass is made of $\sim 25\%$ of ^{4}He, with the rest being made of hydrogen nuclei, which are just free protons. Of course, dark matter is also around, with an abundance which is about 5.5 times as much as regular baryonic matter. But since the Universe is completely dominated by radiation at these early times, dark matter and baryonic matter play no important role in the dynamics of the Universe, that is, their contributions to the Friedmann equation is negligible. However, because the radiation energy density decreases as $1/a^4$ while the matter density goes as $1/a^3$, the importance of matter relative to radiation grows with time.

At these high temperatures, all hydrogen and helium atoms are ionized since there are too many high-energy photons with energies much higher than the atomic binding energy of these atoms. These photons constantly scatter off free electrons via a process called Compton (or Thomson in the nonrelativistic limit) scattering

$$e^- + \gamma \to e^- + \gamma. \tag{1}$$

This process keeps the free electrons and photons in thermal equilibrium at the same temperature. This process also ensures that the *mean free path* (the average distance traveled between two collisions) of photons is very small. This means that the Universe is *opaque* at this epoch. Meanwhile, Coulomb scattering between free electrons and protons (or helium nuclei) keeps these constituents in thermal equilibrium at the same temperature

$$p + e^- \to p + e^-. \tag{2}$$

With all of these interactions going on, we say that the visible (baryonic) sector forms a tightly coupled plasma. The neutrinos, which are free streaming through the Universe, do not participate in any of these interactions.

II. THE EPOCH OF MATTER-RADIATION EQUALITY

At some point, the Universe has expanded enough such that the density of matter becomes equal to that of radiation $\rho_{\rm m}(z_{\rm eq}) = \rho_{\rm rad}(z_{\rm eq})$. This can be written as

$$\Omega_{\rm m}(1+z_{\rm eq})^3 = \Omega_{\rm rad}(1+z_{\rm eq})^4 \qquad \Rightarrow \qquad 1+z_{\rm eq} = \frac{\Omega_{\rm m}}{\Omega_{\rm rad}}, \tag{3}$$

where $\Omega_i = \rho_i(t_0)/\rho_{\rm c}$, with $\rho_{\rm c}$ being the critical density of the Universe today $\rho_{\rm c} = 3H_0^2/(8\pi G)$. We have $\Omega_{\rm rad}h^2 = 4.18\times10^{-5}$ and from the latest Planck data, we have $\Omega_{\rm m}h^2 = 0.143$, leading to

$$z_{\rm eq} \simeq 3420. \tag{4}$$

At this redshift, the photon temperature of the Universe was $T_\gamma \sim 1$ eV, and the age of the Universe was about $50,000$ years after the Big Bang. Why do we care about the epoch of matter-radiation equality? We do because it is the epoch at which matter structure can start to grow significantly. This marks the onset of structure formation in our Universe.

III. HELIUM RECOMBINATION

With photon temperatures in the eV range, we are now entering the realm of atomic physics, that is, the typical photon energy is similar to the ionization energies of typical light atoms. The first neutral atoms to form with any significant abundance are helium atoms. With ionization energies of 54.4 eV (He III $\to$ He II) and 24.6 eV (He II $\to$ He I), we might naively expect that the He III $\to$ He II transition occurs when $T_\gamma \sim 54$ eV while the He II $\to$ He I transition occurs at $T_\gamma \sim 25$ eV. However, the fact that $\eta_{\rm b} \sim 6\times10^{-10}$ again plays an important role as there are always many, many photons for each atom that is trying to form, and these photons keep ionizing them. So, we have to wait until T_γ falls *significantly* below the ionization energy of a given electronic state before an electron can safely end up there permanently. We will see how this plays out for hydrogen below, but the He III $\to$ He II transition starts at $z \sim 6000$ ($T_\gamma \sim 1.4$ eV $\ll 54.4$ eV), while the He II $\to$ He I transition starts at $z \sim 2500$ ($T_\gamma \sim 0.6$ eV $\ll 24.6$ eV).

By $z \sim 1500$ ($T_\gamma \sim 0.35$ eV), all helium nuclei have "recombined" into neutral atoms. This leaves a plasma made of an equal number of electrons and protons, which are still tightly coupled (i.e. frequently interacting via Thomson scattering) with photons. The Universe is thus still opaque at these redshifts. The next step is for atomic hydrogen to start forming.

IV. HYDROGEN RECOMBINATION

At $T_\gamma \sim 0.4$ eV, the reaction

$$e^- + p \leftrightarrow \mathrm{H} + \gamma, \tag{5}$$

is still in thermal equilibrium, implying that the chemical potentials satisfy $\mu_p + \mu_e = \mu_H$ (remember that $\mu_\gamma = 0$). Consider the ratio of number densities

$$\left(\frac{n_\mathrm{H}}{n_\mathrm{e} n_\mathrm{p}}\right)_\mathrm{eq} = \frac{g_\mathrm{H}}{g_\mathrm{e} g_\mathrm{p}} \left(\frac{m_\mathrm{H}}{m_\mathrm{e} m_\mathrm{p}} \frac{2\pi}{T}\right)^{3/2} e^{(m_\mathrm{p}+m_\mathrm{e}-m_\mathrm{H})/T}. \tag{6}$$

The factor in the exponential is the well-known atomic binding energy of hydrogen

$$B_\mathrm{H} \equiv m_\mathrm{p} + m_\mathrm{e} - m_\mathrm{H} = 13.6\,\mathrm{eV}. \tag{7}$$

In the prefactor, we can neglect the difference between m_H and m_p and set $m_\mathrm{H} \approx m_\mathrm{p}$. The spin degeneracy factor of hydrogen is $g_\mathrm{H} = 1 + 3 = 4$ since adding two spin-1/2 particles (the spins of the proton and the electron) results in both a singlet and triplet states. We also have $n_\mathrm{p} = n_\mathrm{e}$ since the Universe is electrically neutral. We thus get

$$\left(\frac{n_\mathrm{H}}{n_\mathrm{e}^2}\right)_\mathrm{eq} \simeq \left(\frac{2\pi}{m_\mathrm{e} T}\right)^{3/2} e^{B_\mathrm{H}/T}. \tag{8}$$

It is useful to introduce the free electron fraction,

$$X_e \equiv \frac{n_\mathrm{e}}{n_\mathrm{p} + n_\mathrm{H}} \simeq \frac{n_\mathrm{e}}{n_\mathrm{b}}, \tag{9}$$

where n_b is the baryon number density. We have neglected the small number density of helium in the last equality. The baryon number density can be written as

$$n_\mathrm{b} = \eta_\mathrm{b} n_\gamma = \eta_\mathrm{b} \frac{2\zeta(3)}{\pi^2} T^3. \tag{10}$$

With the above definition, we get

$$\frac{1 - X_e}{X_e^2} = \left(\frac{n_\mathrm{H}}{n_\mathrm{e}^2}\right) n_\mathrm{b}, \tag{11}$$

such that we can write

$$\left(\frac{1 - X_e}{X_e^2}\right)_\mathrm{eq} = \frac{2\zeta(3)}{\pi^2} \eta_\mathrm{b} \left(\frac{2\pi T}{m_\mathrm{e}}\right)^{3/2} e^{B_\mathrm{H}/T}. \tag{12}$$

The above is called the *Saha equation*. It describes the equilibrium behavior of the ionization fraction $X_e(T)$. Much like in our discussion of the early deuterium abundance, the factor of $\eta_\mathrm{b} \sim 6 \times 10^{-10}$ in the right-hand side means that $X_e \simeq 1$ (i.e. hydrogen is completely ionized) until $T \ll B_\mathrm{H}$. As you will show in the homework, the temperature of the Universe when 90% of the hydrogen atoms have recombined is $T_\mathrm{rec} \approx 0.3$ eV ($z_\mathrm{rec} \approx 1300$).