PHYS480/581 Cosmology
Hydrogen Recombination

(Dated: November 7, 2022)

I. OUT-OF-EQUILIBRIUM HYDROGEN RECOMBINATION

Last time we introduce the Saha equation describing the equilibrium ionization fraction

$$\left(\frac{1-X_e}{X_e^2}\right)_{\rm eq} = \frac{2\zeta(3)}{\pi^2}\eta_{\rm b}\left(\frac{2\pi T}{m_{\rm e}}\right)^{3/2} e^{B_{\rm H}/T}. \tag{1}$$

Much like in our discussion of the early deuterium abundance, the factor of $\eta_{\rm b} \sim 6\times 10^{-10}$ in the right-hand side means that $X_e \simeq 1$ (i.e. hydrogen is completely ionized) until $T \ll B_{\rm H}$. As you will show in the homework, the temperature of the Universe when 90% of the hydrogen atoms have recombined is $T_{\rm rec} \approx 0.3$ eV ($z_{\rm rec} \approx 1300$).

The Saha equation is useful to predict the temperature at which the process we call hydrogen "recombination" starts. However, it is not appropriate to compute the detailed evolution of $X_e(T)$. This is because as soon as recombination starts, the process goes out of equilibrium and we thus need to solve a *Boltzmann equation* for the evolution of X_e. This equation takes the form

$$\frac{dX_e}{dt} = \langle\sigma_{\rm rec}v\rangle\left\{(1-X_e)\left(\frac{m_{\rm e}T}{2\pi}\right)^{3/2} e^{-B_{\rm H}/T} - X_e^2 n_{\rm b}\right\}, \tag{2}$$

where $\langle\sigma_{\rm rec}v\rangle$ is the thermally averaged recombination rate. The first term in the large curly bracket represent the photoionization process $\mathrm{H}+\gamma \to e^- + p$, which destroys neutral hydrogen. It is proportional to $1-X_e$ since this determine the abundance of neutral hydrogen at that time. This terms comes with an overall positive sign since it increases X_e (i.e. it makes the Universe more ionized). The second term in the curly bracket is the recombination term $e^- + p \to \mathrm{H}+\gamma$. It is proportional to X_e^2 since one needs to bring an electron and a proton together to form a neutral atom. This term is overall negative since it decreases X_e (i.e. it makes the Universe more neutral). Note that setting the whole term in the large curly bracket to zero yields the Saha equation written above.

Using the fact that

$$dt = \frac{da}{aH}, \tag{3}$$

the above can be rewritten as

$$\frac{dX_e}{da} = \frac{\langle\sigma_{\rm rec}v\rangle n_{\rm b}}{aH}\left\{\frac{(1-X_e)}{n_{\rm b}}\left(\frac{m_{\rm e}T}{2\pi}\right)^{3/2} e^{-B_{\rm H}/T} - X_e^2\right\}. \tag{4}$$

At $T \gtrsim 0.4$ eV, the prefactor

$$\frac{\langle\sigma_{\rm rec}v\rangle n_{\rm b}}{aH} \tag{5}$$

is quite large, and the only solution to the above differential equation is to set the large curly bracket to zero, i.e. the Saha equation. However, as soon as recombination starts, this prefactor becomes small, that is, the recombination rate becomes small compared to the Hubble rate, and the difference between the two terms in the large bracket becomes important. In this case, one needs to solve the differential equation numerically. The thermally averaged recombination rate is

$$\langle\sigma_{\rm rec}v\rangle \simeq 9.78\frac{\alpha^2}{m_{\rm e}^2}\left(\frac{B_{\rm H}}{T}\right)^{1/2}\ln\left(\frac{B_{\rm H}}{T}\right), \tag{6}$$

which is valid for $T < B_{\rm H}$. See Fig. 1 below for the detailed evolution of $X_e(z)$.

The picture presented above is a little simplistic as it assumes direct recombination to the ground state of hydrogen. As was pointed out by Pebbles in 1968, direct recombination to the ground state does not lead to any net decrease of the ionization fraction X_e. This is because the 13.6 eV photon emitted in such a transition has an extremely large cross section to be reabsorbed by another nearby neutral hydrogen atom, due to the resonant absorption of such photon. Instead, recombination has to proceed through excited states, especially the $2s$ and $2p$ states. So the picture is that electrons first recombine to these $n = 2$ excited states and then have to make their way to the ground state. The $2s \to 1s$ transition is technically a forbidden transition, that is, it cannot occur by the emission of a single photon since both initial and final states have zero angular momentum ($\ell = 0$) and photons are spin-1 particles. It can however proceed by the emission of two photons, but this is a very slow process $\Lambda_{2\gamma} = 8.227$ sec^{-1}.

On the other hand, the $2p \to 1s$ transition is an allowed process, which emits a well-known Lyman-α photon. The problem is that the cross section for these photons to be reabsorbed by another neutral atom nearby is very large (again, it is a resonant absorption). Thus, the $2p \to 1s$ transition can only result on a net ground state recombination event if the photon can redshift out of the Lyman-α resonance before encountering another hydrogen atom. This redshifting rate is given by

$$\Lambda_\alpha = \frac{H(3B_{\rm H})^3}{(8\pi)^2 n_{\rm H}}. \tag{7}$$

This atomic physics could be taken into account by multiplying the recombination rate in the Boltzmann equation above by the so-called "Peebles" factor

$$C_{\rm Peebles} = \frac{\Lambda_\alpha + \Lambda_{2\gamma}}{\Lambda_\alpha + \Lambda_{2\gamma} + \beta^{(2)}} \tag{8}$$

where $\beta^{(2)}$ is the photoionization rate of the $2p$ state, which is given by

$$\beta^{(2)} = \langle \sigma_{\rm rec} v \rangle \left(\frac{m_{\rm e} T}{2\pi} \right)^{3/2} e^{-B_{\rm H}/4T}. \tag{9}$$

As long as $\beta^{(2)} \gg \Lambda_\alpha, \Lambda_{2\gamma}$, the Peebles factor is small and this slows recombination down. Only when enough $n = 2$ states can form can recombination proceed in earnest.

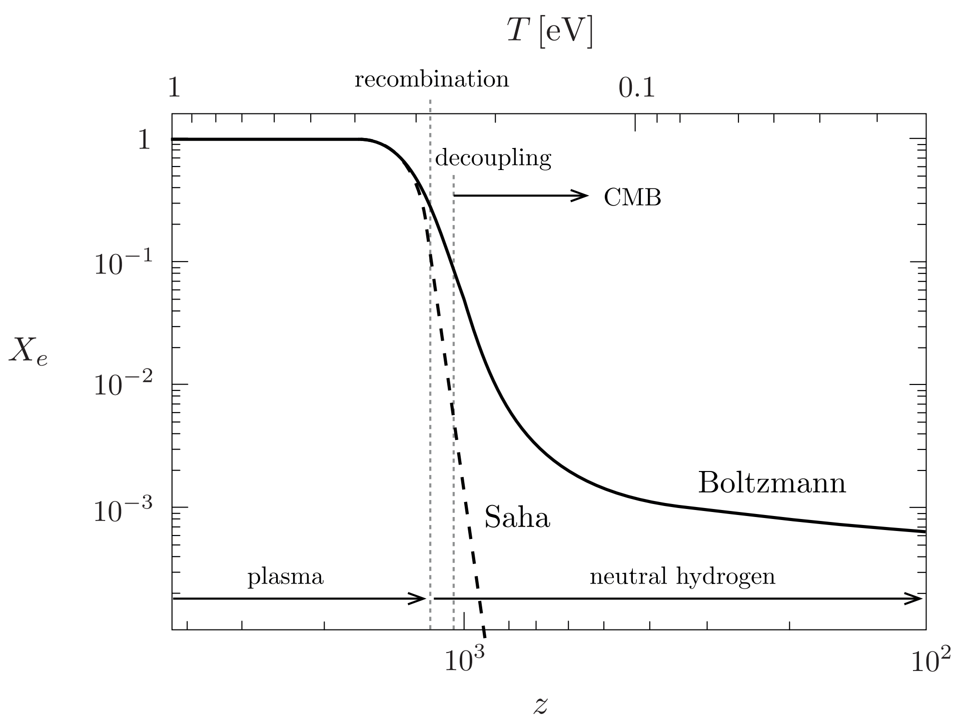

FIG. 1. The ionization history of the Universe. Figures from Baumann (2022).