PHYS480/581 Cosmology
Summary and Acceleration

(Dated: October 3, 2022)

I. THE COSMOLOGICAL PRINCIPLE AND THE METRIC

So far, we have argued that the large-scale homogeneity and isotropy embedded in the Cosmological Principle determine the form of the spacetime metric to be

$$ds^2 = -dt^2 + a(t)^2 \left[d\chi^2 + S_k^2(\chi) d\Omega^2\right], \tag{1}$$

where the spatial coordinates are comoving, $a(t)$ is the scale factor describing the expansion, and where

$$S_k(\chi) = \begin{cases} \frac{1}{\sqrt{\Omega_K} H_0} \sinh\left(\sqrt{\Omega_K} H_0 \chi\right) & \text{if } \Omega_K > 0, \\ \chi & \text{if } \Omega_K = 0, \\ \frac{1}{\sqrt{|\Omega_K|} H_0} \sin\left(\sqrt{|\Omega_K|} H_0 \chi\right) & \text{if } \Omega_K < 0. \end{cases} \tag{2}$$

This metric is very important as it allows us to measure *physical* distances in the constantly changing expanding Universe. The three cases for the function $S_k(\chi)$ corresponds to the three possible global spatial geometries allowed by the Cosmological Principle: either open (hyperbolic), flat, or closed (spherical) geometries. These are the only possibilities for geometries that have constant curvature (including a vanishing one) everywhere.

II. THE FRIEDMANN EQUATION

The evolution of the scale factor $a(t)$ is given by the Friedmann equation, which always relates the energy content of the Universe to the Hubble rate of expansion $H \equiv \dot{a}/a$. Written in terms of the density parameters $\Omega_i \equiv \rho_i(t_0)/\rho_c$, where $\rho_c = 3H_0^2/(8\pi G)$, it takes the form

$$H^2(z) = H_0^2 \left[\Omega_{\rm rad}(1+z)^4 + \Omega_{\rm m}(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right], \tag{3}$$

where we have written the scale factor as

$$a = \frac{1}{1+z}, \tag{4}$$

where z is redshift, and where have assumed a universe populated by radiation ($\Omega_{\rm rad}$), matter ($\Omega_{\rm m}$), and dark energy (Ω_Λ), while allowing for the presence of spatial curvature (Ω_K). Remember that we have the constraint

$$\Omega_K = 1 - \sum_i \Omega_i, \tag{5}$$

where the sum includes everything except for the curvature term. The different scaling with redshift appearing in the Friedmann equation were gotten by solving the fluid equation

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{6}$$

for components with constant equation of state $w = p/\rho$. This lead to a solution of the form $\rho \propto a^{-3(1+w)}$, which for matter ($w = 0$) gives $\rho_{\rm m} \propto a^{-3}$, for radiation ($w = 1/3$) $\rho_{\rm rad} \propto a^{-4}$, and for dark energy ($w = -1$) $\rho_\Lambda \propto$ const. We have seen that the radiation budget of the Universe is dominated by photons and relativistic neutrinos, with potential small contributions from massless particles beyond the Standard Model. The matter budget is dominated by dark matter, with a small contributions from baryons, and an even smaller contribution from massive neutrinos at late times.

III. AGES AND DISTANCES

The age of the Universe at a given value of the scale factor (or redshift) is given by

$$t(a) = \int_0^t dt' = \int_0^a \frac{da'}{a'H(a')} = \int_{z(a)}^{\infty} \frac{dz'}{(1+z')H(z')}. \tag{7}$$

These ages are always proportional to the Hubble time $t_H = 1/H_0$. A very important distance in our expanding Universe is the comoving distance that a photon has travelled from some time of emission in the past to today. Since photons always travel on null paths ($ds^2 = 0$), that distance is

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')} = \int_0^{z(a)} \frac{dz'}{H(z')}. \tag{8}$$

Note that this χ is the same χ coordinate appearing in Eq. (1). In our Universe, we have two main ways of measuring distances. First, if we happen to know the *physical* size l of some object or feature in the sky and can measure its apparent angular size θ and its redshift with observations, we can extract the angular diameter distance

$$d_{\rm A}(z) = \frac{l}{\theta}. \tag{9}$$

The nice thing is that this observed distance can be compared to its theoretical prediction within any cosmological model, which is given by

$$d_{\rm A}(z) = \frac{S_k(\chi(z))}{1+z}, \tag{10}$$

with χ given by Eq. (8) above. For a spatially flat cosmology ($\Omega_K = 0$), this is simply $d_{\rm A}(z) = \chi(z)/(1+z)$. Another way to measure distances is to measure the apparent magnitude m and redshift z of a source of known intrinsic luminosity (absolute magnitude M). Its luminosity distance is then given by

$$d_{\rm L}(z) = 10^{\frac{m-M-25}{5}}\ {\rm Mpc}. \tag{11}$$

This measurement can then be compared with the theoretical prediction for the luminosity distance in any cosmological model, which is given

$$d_{\rm L}(z) = (1+z) S_k(\chi(z)). \tag{12}$$

For a spatially flat cosmology ($\Omega_K = 0$), this is simply $d_{\rm L}(z) = (1+z)\chi(z)$.

IV. DECELERATION PARAMETER

When probing the local Universe, it is sometime useful to consider the Taylor expansion of the scale factor $a(t)$

$$a(t) = a(t_0) + \dot{a}(t_0)(t-t_0) + \frac{1}{2}\ddot{a}(t_0)(t-t_0)^2 + \ldots. \tag{13}$$

Dividing by $a(t_0)$, we obtain

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) - \frac{q_0}{2} H_0^2 (t-t_0)^2 + \ldots, \tag{14}$$

where we have defined

$$q_0 \equiv -\frac{\ddot{a}(t_0)}{a(t_0)H_0^2} = -\frac{\ddot{a}(t_0)a(t_0)}{\dot{a}^2(t_0)}, \tag{15}$$

which is called the deceleration parameter. It is dimensionless. Before the discovery of dark energy, most astrophysicists believed that the universe was decelerating, so the sign convention was chosen such that a universe that is slowing down would have $q_0 > 0$. Since our Universe appears to be accelerating, we are now stuck with a negative q_0! In any case, in simple cosmologies, q_0 admits simple values so it is a useful parameters to consider.

To compute q_0, we need to determine a value for $\ddot{a}(t)$. Taking a time derivative of the Friedmann equation and using the fluid equation, we obtain (see worksheet) the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{16}$$

where ρ and p are the total energy density and pressure of the universe. Thus, to have an accelerating universe ($\ddot{a} > 0$), we need to have $\rho + 3p < 0$, that is,

$$w < -1/3. \tag{17}$$

A universe with just a cosmological constant ($w = -1$) is obviously satisfying this bound, but even a universe with, say, $w = -1/2$, would result in an accelerating universe today.

Equipped with the acceleration equation, we can compute the value of q_0 in different Universes. For example, in a matter dominated Universe, the deceleration parameter is

$$q_0 = \frac{4\pi G}{3H_0^2}\rho_{\rm m}(t_0) = \frac{1}{2}\frac{8\pi G}{3H_0^2}\rho_{\rm m}(t_0) = \frac{1}{2}\frac{\rho_{\rm m}(t_0)}{\rho_{\rm c}} = \frac{\Omega_{\rm m}}{2}. \tag{18}$$