PHYS480/581 Cosmology
Symmetry and the structure of the cosmological metric

(Dated: August 29, 2022)

I. THE FLAT FRIEDMANN-LEMAÎTRE-ROBERTRON-WALKER (FLRW) METRIC

We need our spacetime metric describing our Universe to reflect the cosmological principle, which states that the Universe is, on average, homogeneous (in space) and isotropic. Since three-dimensional Euclidean space is homogeneous and isotropic, we only need a small modification to the Minkowski metric to describe a smooth, expanding Universe. Using spatial cartesian coordinates (which are orthogonal), the most general metric we can write down is

$$ds^2 = -f(t,x,y,z)dt^2 + g(t,x,y,z)dx^2 + h(t,x,y,z)dy^2 + l(t,x,y,z)dz^2, \tag{1}$$

where f, g, h, l are arbitrary functions at this point. The cosmological principle puts strong constraints on these functions. Let's start by considering isotropy. An isotropic universe does not have any preferred direction, which means that moving in the x, y, or z direction should be completely equivalent. The only way to enforce this is to have $f = g = h$, that is

$$ds^2 = -f(t,x,y,z)dt^2 + g(t,x,y,z)\left(dx^2 + dy^2 + dz^2\right). \tag{2}$$

As we will see in a few lectures, this is a little too restrictive. But the above is valid when we have *spatially flat* (Euclidean) three-dimensional geometry. Now considering homogeneity, which states that there is no special point in the Universe, we must have $f = f(t)$ and $g = g(t)$ only. That is f and g cannot depends on the coordinates x, y, z. It is customary to denote the function g as $a^2(t)$. We are thus left with

$$ds^2 = -f(t)dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{3}$$

As with any coordinate system, we are always free to redefine our coordinates. For instance, it is customary to let $\sqrt{f(t)}dt \to dt$, that is, absorb the function f into the definition of our time coordinate. With this, we obtain the *flat* Friedmann-Lemaître-Robertron-Walker metric, written in cartesian coordinates

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{4}$$

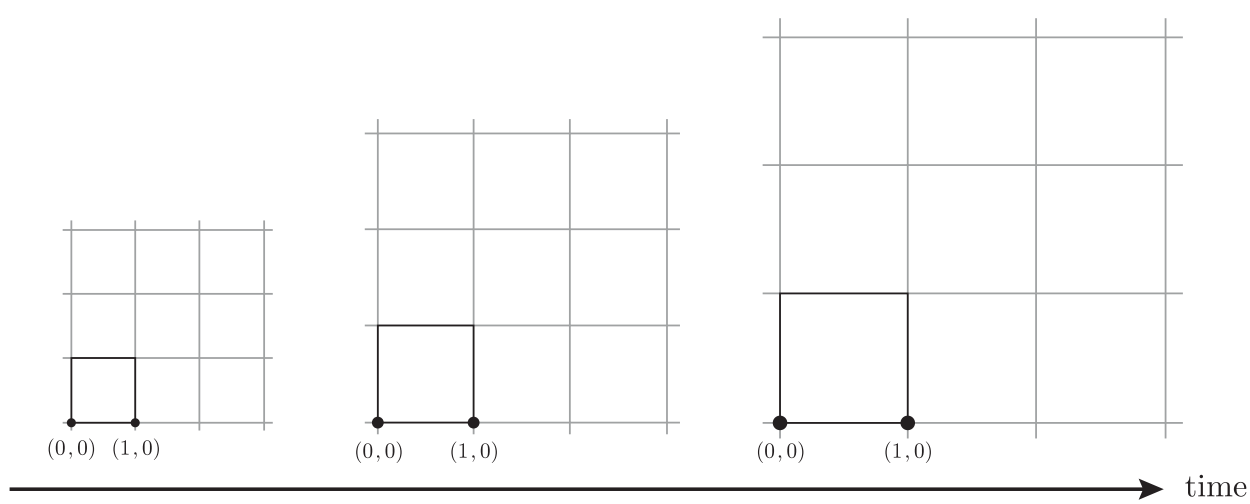

FIG. 1. The comoving distance between points on an imaginary coordinate grid remains constant as the universe expands. The physical distance is proportional to the comoving distance times the scale factor $a(t)$ and hence gets larger as time evolves. Image credits: D. Baumann.

Here, the spatial coordinates (x, y, z) are called *comoving coordinates*. These coordinates form a fixed grid on which the *coordinate* distance between any point is always the same. However, the *physical* distance between points is modulated by $a(t)$ (see Fig. 1), which in our Universe, is an increasing function of time, as we will see. Objects separated by a fixed comoving distance $r_{\rm com}$ have a physical separation $r_{\rm phys}$ given by

$$r_{\rm phys}(t) = a(t) r_{\rm com}. \tag{5}$$

It is customary to define $a(t_0) = 1$, where t_0 denotes the present time. This means that today we have $r_{\rm phys} = r_{\rm com}$. With this convention, when we quote the comoving distance to an object, it corresponds to its physical distance today. If $a(t)$ is an increasing function of time (as it is in our Universe), then $a(t) < 1$ for $t < t_0$, and objects at a fixed comoving distance were closer to us in the past.

The function $a(t)$ is called the *scale factor*, as it tells us how physical distances evolve as a function of time in our Universe.

II. THE LINE ELEMENT,HE LIGHT-CONE, AND PROPER TIME.

The four-dimensional line element ds^2 is a physical quantity that any observer, in any inertial frame of reference, will agree on. In general, there are three types of ds^2 element that we can have:

1. $ds^2 < 0$: This is a timelike interval. Causally related events are always separated by a timelike interval.

2. $ds^2 = 0$: This is a null (or lightlike) interval. Particle moving at the speed of light (like photons) always move on null paths.

3. $ds^2 > 0$: This is a spacelike interval. No causally related events can be separated by a spacelike interval.

The spacetime element ds^2 is also important since it can be used to compute the time that any observer will measure on their watch as they move through spacetime. This elapsed time, called the *proper* time is simply given by $d\tau = \sqrt{-ds^2}$.