PHYS480/581 Cosmology
e^+e^- annihilation, neutrino decoupling, and entropy conservation

(Dated: October 18, 2022)

I. NEUTRINO DECOUPLING

As we saw last time, the weak interaction (mediated by the W and Z boson) can no longer maintain the neutrinos in kinetic equilibrium with the rest of the plasma (made of e^-, e^+, and γ) at temperatures below $T \sim 1$ MeV. This leaves the neutrinos on their own, with a distribution function reflecting what it was at the time of decoupling

$$f_\nu(p) = \frac{1}{e^{p/T_\nu} + 1}, \tag{1}$$

that is, the relativistic Fermi-Dirac distribution. Since neutrinos are no longer interacting with any other particles, no particle collision can change this distribution from that point onward. Only the expansion of the Universe acts to cool the neutrino temperature as $T_\nu \propto 1/a$, where a is the scale factor of the Universe. Maintaining the invariance of $f_\nu(p)$ requires that the momentum $p \propto 1/a$. This turns our to be always true for any particle (relativistic or not), but it is easy to see this scaling for, say, photons, for which $p = E \propto 1/\lambda$, where the wavelength $\lambda \propto a$ is stretched by the expansion of the Universe (causing what we called redshift). Since the distribution function is now fixed, quantities derived from it, such as the neutrino number density, are simple to calculate

$$n_\nu = 2\int \frac{d^3p}{(2\pi)^3}\frac{1}{e^{p/T_\nu} + 1} = \frac{3}{4}\frac{2\zeta(3)}{\pi^2}T_\nu^3, \tag{2}$$

for one species of neutrinos. This scales as $n_\nu \propto 1/a^3$ (since $T_\nu \propto 1/a$) as we expect from volumetric dilution from the expansion. The key question is then, what is T_ν? Before e^+e^- annihilation, the answer is simply $T_\nu = T_\gamma$, where T_γ is the photon temperature. However, after e^+e^- annihilate and dump their energy in the photon bath, the photons gain some energy compared to the neutrinos. Since the neutrinos receive none of this energy (since they are decoupled from the rest of the plasma), this leads to $T_\gamma > T_\nu$ after e^+e^- annihilation ends. To compute this temperature difference, we need to discuss the conservation of entropy in an expanding universe.

II. CONSERVATION OF ENTROPY

The second law of thermodynamics states that the entropy S is always either constant or increasing $dS \geq 0$. If the Universe is in thermal equilibrium as it expands, it turns out (as we will show below) that the inequality is *saturated* and the entropy of the Universe is *constant*. In thermodynamics language, this means that we can consider the expansion of the Universe as an *adiabatic* (reversible) process. To see this, we first need to derive an expression for the entropy of the Universe in terms of quantity we already know like the energy density ρ and the pressure P. To do so, we will need the following relation

$$\frac{\partial P}{\partial T} = \frac{\rho + P}{T}, \tag{3}$$

which is valid in thermal equilibrium with $\mu/T \ll 1$. This expression is easy to generalize for large chemical potentials, but we will neglect the chemical potential here for simplicity. Now, starting from the first law of thermodynamics (assuming again that $\mu/T \ll 1$)

$$TdS = dE + PdV \tag{4}$$

where E is the energy and V is the volume. Writing $E = \rho V$, we get

$$\begin{aligned} dS &= \frac{1}{T}\left(d(\rho V) + PdV\right) \\ &= \frac{1}{T}\left(d\left[(\rho + P)V\right] - VdP\right) \\ &= \frac{1}{T}d\left[(\rho + P)V\right] - \frac{V}{T^2}\left(\rho + P\right)dT \\ &= d\left[\frac{\rho + P}{T}V\right], \end{aligned} \tag{5}$$

which shows that the entropy of the Universe is simply

$$S = \frac{\rho + P}{T} V. \tag{6}$$

As usual in cosmology, we will be interested in the *entropy density* (or entropy per unit volume) s,

$$s \equiv \frac{S}{V} = \frac{\rho + P}{T}. \tag{7}$$

The evolution of s is given by

$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{T}\frac{d}{dt}(\rho + P) - \frac{(\rho + P)}{T^2}\frac{dT}{dt} \\ &= \frac{1}{T}\left(\frac{d\rho}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt} - \frac{\rho + P}{T}\frac{dT}{dt}\right) \\ &= \frac{-3H(\rho + P)}{T} = -3Hs, \end{aligned} \tag{8}$$

where we have used Eq. (3) in going from the second to third line, and where we have used the fluid equation $\dot{\rho} + 3H(\rho + P) = 0$ in the third line. This means that the entropy density obeys the following equation

$$\frac{ds}{dt} + 3Hs = \frac{1}{a^3}\frac{d}{dt}\left(a^3 s\right) = 0, \tag{9}$$

which means that $a^3 s$ =constant in an expanding universe. This makes sense since $s = S/V$ and $V \propto a^3$, but it can only be true if S = constant. This shows that the total entropy is constant in an expanding Universe.

III. ENTROPY IN OUR UNIVERSE

Now that we know the definition of entropy density given in Eq. (7), we can compute its value

$$s = \sum_i \frac{\rho_i + P_i}{T_i}. \tag{10}$$

This sum is always completely dominated by relativistic particles (i.e. radiation) since non-relativistic particles in our Universe have either exponentially suppressed abundances ($n \propto e^{-m/T}$), or have number densities that are orders of magnitude smaller than that of photons. For radiation, we have $P_i = \rho_i/3$, so s is given by

$$s = \frac{4}{3}\sum_i \frac{\rho_i}{T_i} = \frac{4}{3}\frac{\pi^2}{30}\left(\sum_{i\in\text{bosons}} g_i T_i^3 + \frac{7}{8}\sum_{j\in\text{fermions}} g_j T_j^3\right) = \frac{2\pi^2}{45} g_{*S}(T) T^3, \tag{11}$$

where

$$g_{*S}(T) = \sum_{i\in\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{j\in\text{fermions}} g_j \left(\frac{T_j}{T}\right)^3 \tag{12}$$

is the *effective number of degrees of freedom in entropy.* We see that if all particle in the Universe are in thermal equilibrium at the same temperature, we have $g_{*S}(T) = g_*(T)$. This is true before neutrino decoupling, but not after. Conservation of entropy $a^3 s$ = constant implies that

$$sa^3 = \frac{2\pi^2}{45} g_{*S}(T) T^3 a^3 = \text{constant} \rightarrow T \propto g_{*S}^{-1/3} a^{-1}. \tag{13}$$

When g_{*S} is constant, we retrieve the standard result that $T \propto 1/a$. However, when g_{*S} is changing T no longer decays as $1/a$, but instead decays *slower.* This reflects the fact that when g_{*S} changes, particles are disappearing from the plasma and transferring their entropy and energy to other particles in the plasma, causing their temperature to decay less fast. We now want to apply the concept of entropy conservation to the process of e^+e^- annihilation.

IV. NEUTRINO TEMPERATURE AFTER e^+e^- ANNIHILATION

We are now ready to use entropy conservation to compute the temperature ratio T_ν/T_γ after e^+e^- annihilation has played out. First, let's consider the entropy density of the Universe at time t_1 (scale factor $a_1 = a(t_1)$ and temperature T_1) before e^+e^- annihilation

$$s(a_1) = \frac{2\pi^2}{45}\left(2 + \frac{7}{8}(2+2+3+3)\right)T_1^3 = \frac{43\pi^2}{90}T_1^3. \tag{14}$$

After the completion of e^+e^- annihilation, the entropy density at time t_2 (scale factor $a_2 = a(t_1)$) is

$$s(a_2) = \frac{2\pi^2}{45}\left(2T_\gamma^3(a_2) + \frac{7}{8}(3+3)T_\nu^3(a_2)\right), \tag{15}$$

where we have now allowed for photons and neutrinos to have different temperatures since photons get the entropy and energy from the annihilating e^+e^- while neutrinos do not. Entropy conservation implies that $s(a_1)a_1^3 = s(a_2)a_2^3$, which leads to

$$\begin{aligned} \frac{43\pi^2}{90}T_1^3a_1^3 &= \frac{2\pi^2}{45}\left(2T_\gamma^3(a_2) + \frac{21}{4}T_\nu^3(a_2)\right)a_2^3 \\ \frac{43}{2}T_1^3a_1^3 &= 4\left(T_\gamma^3(a_2) + \frac{21}{8}T_\nu^3(a_2)\right)a_2^3 \\ \frac{43}{2}T_1^3a_1^3 &= 4\left(\left(\frac{T_\gamma(a_2)}{T_\nu(a_2)}\right)^3 + \frac{21}{8}\right)T_\nu^3(a_2)a_2^3. \end{aligned} \tag{16}$$

Now, neutrinos receive no energy from from e^+e^- annihilation so their temperature always scales as $1/a$, meaning that $a_1T_1 = a_2T_\nu(a_2)$. We are thus left with

$$\frac{43}{8} - \frac{21}{8} = \frac{22}{8} = \frac{11}{4} = \left(\frac{T_\gamma(a_2)}{T_\nu(a_2)}\right)^3, \tag{17}$$

implying that the temperature ratio after e^+e^- annihilation is

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}, \tag{18}$$

so the neutrinos are indeed colder than the photons. After this point, both photon and neutrino temperatures scale as $1/a$ so this ratio stays constant for the remainder of the history of the Universe. The detailed evolution of the temperatures through e^+e^- annihilation is given in Fig. 1 below.

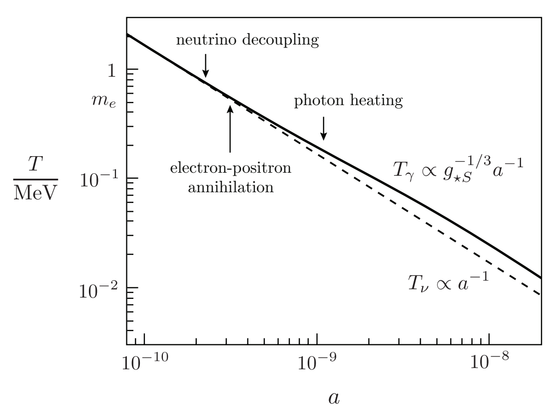

FIG. 1. Evolution of the neutrinos and photon temperatures as a function of scale factor. Figure taken from Baumann (2022).