PHYS480/581 Cosmology
Growth of structure in our Universe

(Dated: November 28, 2022)

I. THE INHOMOGENEOUS UNIVERSE

Last week, we discussed several candidates for what dark matter could be. To be successful, a candidate must behave like *cold dark matter* (CDM) on length scales relevant to cosmology. What does it mean to "behave" like CDM? All we have discussed so far is that the dark matter energy density redshift as a^{-3}, where a is the scale factor. To make further progress in understanding why we need something like CDM, we need to move away from the homogeneous and isotropic Universe we have been discussing so far. In other words, we need to consider *inhomogeneities* and how they evolve in time as the Universe expands.

In an inhomogeneous universe, the energy density of the different constituents of the Universe are no longer just functions of time, but also functions of space, that is, $\rho_i = \rho_i(\mathbf{r}, t)$, where the index i runs over the constituents of the Universe. It is usually convenient to denote the spatial average of the energy density as $\bar{\rho}_i(t)$, which is just a function of time. Basically, these spatial averages are the energy densities we have been dealing with so far in this class (i.e. these enter the right-hand side of the Friedmann equation). With the help of $\bar{\rho}_i$, we can define the dimensionless density contrast δ_i as

$$\delta_i(\mathbf{r}, t) \equiv \frac{\rho_i(\mathbf{r}, t) - \bar{\rho}_i(t)}{\bar{\rho}_i(t)}. \tag{1}$$

At early times, the Universe is nearly homogeneous and δ_i would generally be very small. This means that we can accurately describe the evolution of δ_i using *linear* equations of motion at early times, which simplifies the computation significantly. Here, we would like to develop some understanding of how the matter density contrast δ_{m} grows from nearly zero in the early Universe to very large values describing all the rich inhomogeneous structure we observe today. This growth is largely driven by *gravitational instabilities.*

II. GRAVITATIONAL INSTABILITY

As illustrated in Fig. 1, the assembly of structure in our Universe is governed by the competition between gravitational accretion and pressure repulsion. For structure to form and grow, gravity must be able to overcome this pressure. Unfortunately, many constituents of our Universe have a lot of pressure at early times, and it is difficult to assemble structure using these. Let's briefly review the different constituents of our Universe prior to the epoch of recombination and determine whether gravity or pressure dominates their evolution:

- **Photons:** Photons are always relativistic (they are massless) and thus always have large pressure $P_\gamma = \rho_\gamma/3$. Pressure always dominates over gravity for photons so it is impossible to assemble gravitationally bound structure out of photons.

- **Baryonic matter:** Baryons interact extremely efficiently (via Thomson scattering of free electrons) with photons prior to the epoch of recombination. They thus inherit the pressure of the photons at early times, and baryons cannot form gravitationally bound structure before the epoch of recombination. After recombination, baryons can start forming structure.

- **Neutrinos:** They are ultra-relativistic until late times and thus have have a significant amount of pressure over a significant fraction of the history of the Universe. Therefore, pressure dominates over gravity as long as they are relativistic. Once they become non-relativistic (if it happens), they can start forming gravitationally bound structures. This typically happens in the late Universe.

- **Dark matter:** If dark matter has no pressure even early on, it can start forming gravitationally bound structure right away, even in the radiation-dominated era. Here, gravity always wins. The fact that there were structure in the Universe prior to the epoch of recombination is probably the strongest evidence we have for the existence of dark matter.

Looking at the list above, in a Universe with just baryons, photons, and neutrinos, structure would not be able to start forming until after recombination. Dark matter is necessary to kickstart structure formation at much earlier epochs. Since detailed cosmological observations require structure formation to begin way before the epoch of recombination, dark matter is thus a mandatory component of our Universe.

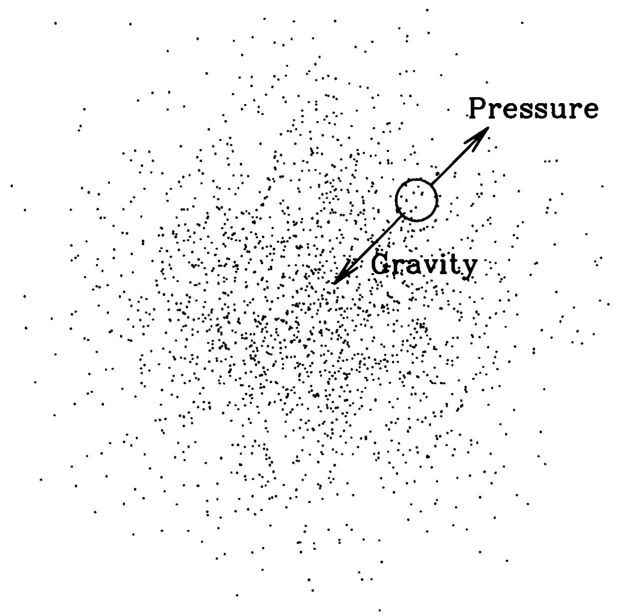

FIG. 1. Gravitational instability. Mass near an overdense region is attracted to the center by gravity but repelled by pressure. If the region is dense enough, gravity wins and the overdensity grows with time. Figure from Dodelson (2003).

How does gravitational instability for non-relativistic matter work? Since gravity is always attractive, any small deviation from pure matter homogeneity will create a force on nearby matter, accelerating it toward the initial perturbation. This makes the fluctuations more massive, which in turns attract more nearby matter, and so on. This is a runaway process, hence the word *instability*. Mathematically in a Minkowski spacetime, we can write this as

$$\ddot{\delta}_{\rm m} - \nabla^2\Phi = 0, \tag{2}$$

where Φ is the Newtonian gravitational potential, which obeys the Poisson equation

$$\nabla^2\Phi = 4\pi G\bar{\rho}_{\rm m}\delta_{\rm m}, \tag{3}$$

where here we have assumed that only matter is present. Combining these two equations, we get

$$\ddot{\delta}_{\rm m} - 4\pi G\bar{\rho}_{\rm m}\delta_{\rm m} = 0, \tag{4}$$

which admits exponentially growing solutions if $\bar{\rho}_{\rm m}$ is approximately. As we will see, this exponential growth will turn into much slower growth once the expansion of the Universe is taken into account.

III. GROWTH OF STRUCTURE IN AN EXPANDING UNIVERSE

Similar to our discussion around inflation, the presence of the Universe's expansion will introduce a *friction* term in the growth of structure equation, that is

$$\ddot{\delta}_{\rm m} + 2H\dot{\delta}_{\rm m} - \nabla^2\Phi = 0, \tag{5}$$

where H is the Hubble rate. As should be apparent from this equation, whether a matter density fluctuation $\delta_{\rm m}$ can grow or not will depend on the Hubble rate and the gravitational potential. Since these behave differently depending what is the dominant energy component of the Universe (i.e. radiation, matter, dark energy, etc.), we will get different behavior in the different era of the Universe.

A. Radiation domination

As we argued above, radiation cannot form gravitationally bound structure. This implies that the gravitational potential is, to a good approximation, negligible during radiation domination, $\Phi \sim 0$. The above equation thus simplifies to

$$\ddot{\delta}_{\rm m} + 2H\dot{\delta}_{\rm m} \approx 0. \tag{6}$$

In the worksheet you will show that the growing solution of this equation is $\delta_{\rm m} \propto \ln a$. So, while photons and neutrinos are dominating the overall energy density of the Universe, dark matter fluctuations can quietly start growing in the background. While logarithmic growth is rather slow compared to the power-law growth that will happen during matter domination (see next time), it does give structure formation a head start that is very important in explaining the amount of structure we observe in the Universe today.