PHYS480/581 Cosmology
Growth of structure in our Universe II

(Dated: November 29, 2022)

I. GROWTH OF STRUCTURE IN AN EXPANDING UNIVERSE

Last time, we saw that non-relativistic matter fluctuations $\delta_{\rm m}$ in an expanding Universe evolve according to the following equation

$$\ddot{\delta}_{\rm m} + 2H\dot{\delta}_{\rm m} - \nabla^2\Phi = 0, \tag{1}$$

where $H = \dot{a}/a$ is the Hubble rate and Φ is the gravitational potential. An overhead dot denotes a (coordinate) time derivative. During radiation domination, we argued that $\Phi \sim 0$ since it is impossible to form gravitationally bound structures out of radiation. This meant that the growing solution to the above equation in radiation domination is $\delta_{\rm m} \propto \ln a$. Thus, dark matter can start forming structure (albeit slowly) even during radiation domination. This gives structure formation a head start that is very important in explaining the amount of structure we observe in the Universe today.

A. Matter domination

While matter structure can slowly grow during radiation domination, most of the growth of structure in our Universe occurs during matter domination. For the purpose of this discussion, we will focus our attention entirely on dark matter and neglect the (relatively small) amount of baryons in the Universe. This simplification sidesteps the question of the changing baryonic pressure support, which is significant before the epoch of recombination, but negligible after that time.

Since matter can actually form gravitationally bound structures, the gravitational potential is not negligible during matter domination. It obeys as usual the Poisson equation

$$\nabla^2\Phi = 4\pi G\bar{\rho}_m\delta_{\rm m}, \tag{2}$$

where $\bar{\rho}_{\rm m}$ is the mean matter density in the Universe (which scales as $\bar{\rho}_{\rm m} \propto a^{-3}$). Substituting this in Eq. (1), we obtain

$$\ddot{\delta}_{\rm m} + 2H\dot{\delta}_{\rm m} - 4\pi G\bar{\rho}_m\delta_{\rm m} = 0, \tag{3}$$

As you will show in the worksheet, the growing solution to this equation is $\delta_{\rm m} \propto a$. This linear growth (in the scale factor) is much faster than the logarithmic growth during radiation domination, and thus most of the growth of structure in our Universe occurs during matter domination.

What about the gravitational potential during matter domination? Going back to Eq. (2) and substituting $\delta_{\rm m} \propto a$ and $\bar{\rho}_{\rm m} \propto a^{-3}$ in the right-hand side, we obtain

$$\nabla^2\Phi \propto a^{-2}. \tag{4}$$

This seems to imply that the gravitational potential is time-dependent. However, the Laplacian operator ∇^2 here is written in *physical* coordinates. Remembering that physical and comoving coordinates are schematically related via

$$r_{\rm phys}(t) = a(t)r_{\rm com}, \tag{5}$$

where $r_{\rm com}$ is the comoving distance, we have that

$$\nabla^2_{\rm com} = a^2\nabla^2_{\rm phys}, \tag{6}$$

where $\nabla^2_{\rm com}$ is the Laplacian operator written in comoving coordinates, while $\nabla^2_{\rm phys}$ is the Laplacian in physical coordinates. Using this, we can rewrite Eq. (2) as

$$\nabla^2_{\rm com}\Phi = 4\pi G a^2\bar{\rho}_m\delta_{\rm m}. \tag{7}$$

Substituting $\delta_{\rm m} \propto a$ and $\bar{\rho}_{\rm m} \propto a^{-3}$ in the right-hand side, we now get

$$\nabla^2_{\rm com}\Phi \propto \text{constant}. \tag{8}$$

This means that once written as a function of comoving coordinates, the gravitational potential is a constant function of time (it can still vary in space).

The picture that thus emerges during matter domination is of dark matter fluctuations growing linearly with the scale factor while the gravitational potential is constant in time. This is quite different than what happens during radiation domination, where matter fluctuations grow logarithmically and the gravitational potential is essentially vanishing. These different regimes are illustrated in Fig. 1 where we see the logarithmic growth during radiation domination, and the linear growth during matter domination.

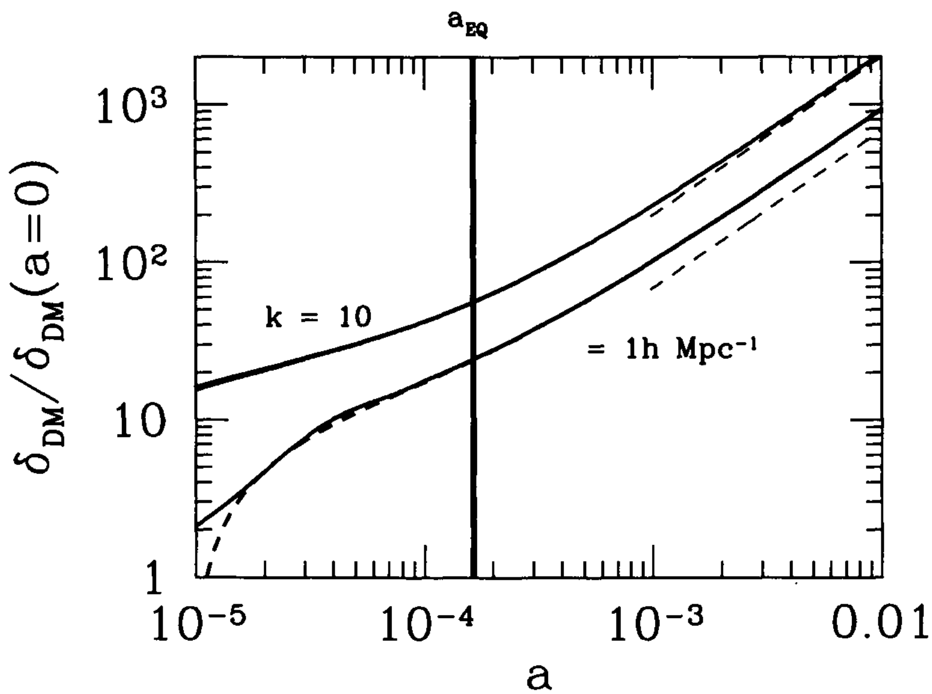

FIG. 1. Evolution of dark matter density fluctuations as a function of the scale factor (normalized to an arbitrary initial value at $a = 0$) for two different spatial extents of fluctuations, characterized by their comoving wavenumber k. Focusing on the mode with $k = 10h$ Mpc^{-1}, we see the logarithmic growth before matter-radiation equality ($a < a_{\rm eq}$) and the linear growth emerging for $a \gg a_{\rm eq}$ (the dashed line for large a show $\delta_{\rm DM} \propto a$). Figure from Dodelson (2003).

B. Remarks

There are a few important remarks about our derivation of the different growth regime discussed above.

- The discussion presented here is only valid for $\delta_{\rm m} \ll 1$, that is, for small density fluctuations. This is because first-order perturbation theory was used to derive equations like Eq. (1). For $\delta_{\rm m} \sim 1$, other terms that we have neglected here will become important and dramatically affect the evolution of $\delta_{\rm m}$. We say that the evolution of $\delta_{\rm m}$ enters the *non-linear* regime when $\delta_{\rm m}$ approaches unity. While the linear regime discussed here is very useful to describe the large-scale distribution of galaxies, it cannot be used to understand how a galactic dark matter halo forms.

- The discussion above applies to fluctuations that have spatial extent L that are much smaller than the size of the Hubble horizon, $L \ll H^{-1}$. For fluctuations of size $L \sim H^{-1}$, the Poisson equation we used to describe the gravitational potential no longer applies and other terms becomes important in describing of the gravitational potential. Because of this, as the Hubble horizon expands and more and more fluctuation modes become causal, modes entering the horizon during radiation domination will not immediately follow the logarithmic solution; there will be a transient behavior for some time before the fluctuation settle in the logarithmic solution. This can be seen in Fig. 1 for the mode with wavenumber $k = 1h$ Mpc^{-1}.

- Close to matter-radiation equality, the Universe is neither completely dominated by radiation nor matter. This means that there will be a transient period close to matter-radiation equality for which fluctuations are growing faster than $\ln a$ but slower than linear in the scale factor. Obtaining the exact behavior is this regime requires solving the differential for $\delta_{\rm m}$ numerically.
- At very late times $z \lesssim 0.5$, dark energy becomes dynamically important and the Universe is no longer purely matter dominated. Thus, matter fluctuations on large scales no longer growth linearly with the scale factor at late times, but slightly slower. We will show in the homework that matter fluctuations start decaying during dark energy domination.