PHYS 480/581
Cosmology

Homework Assignment 1
Due date: Wednesday September 21 2022, in class

Question 1 (3 points).

Our current universe appears to be dominated by a cosmological constant. Compute the age of our universe assuming that today (when the Hubble expansion rate is $H_0 = 70$ km/s/Mpc) 70% of the energy is in the form of the cosmological constant and 30% is in the form of cold matter.

Question 2 (2 points).

As you saw in the previous problem, the age of the Universe is tightly related to the Hubble constant H_0. However, in the far future, once the cosmological constant *completely* dominates the energy density of our Universe, this relationship will break down. Show that a far-future civilization will determine the age of the Universe to be infinite, independent of the value of H_0. What does this tell you about the symmetry structure of this Universe? Does it have more symmetry as compared to a standard FLRW universe dominated by nonrelativistic matter?

Question 3 (6 points).

Let's consider the flat FLRW metric

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2], \tag{1}$$

where $a(t)$ is the scale factor. For this problem, assume a realistic universe filled with matter, radiation, and a cosmological constant, and use the following values of the cosmological parameters: $H_0 = 70.4$ km/s/Mpc, $\Omega_{\rm r} = 0.000084$, $\Omega_{\rm m} = 0.272$, and $\Omega_\Lambda = 0.728$. Here, we shall refer to epochs in the evolution of the Universe in terms of their *redshift* z, which is related to the scale factor by $a(t) = 1/(1+z)$.

(a) Using the fact that photons always travel on null trajectories ($ds^2 = 0$), compute the total comoving distance that a photon will travel from the Big Bang at $t = 0$ to the epoch of recombination at redshift $z = 1090$.

(b) Now compute the total comoving distance that a photon will travel from the epoch of recombination to the present time ($z = 0$).

(c) Divide your answer from part (a) by that from part (b). Using a diagram, show that this ratio is the maximum angle separating photons that were in causal contact in the distant past, according to the metric above. Express this angle in degrees. Now, we observe that cosmic microwave background photons (which were emitted at the epoch of recombination at $z = 1090$) from opposite points in the sky (i.e. points separated by 180 degrees) to have the same temperature to a high accuracy. Do you see a problem here? Explain.

Question 4 (10 points).

Imagine a spatially-flat homogeneous and isotropic expanding universe filled *exclusively* with a fluid with an equation of state $w = 1$ such that $p = \rho$. Let's denote the present-day Hubble expansion rate as H_0 in this universe.

(a) Using conservation of energy in an expanding universe, show that the energy density of this fluid scales with the scale factor of the Universe $a(t)$ as

$$\rho \propto a(t)^{-6}. \tag{2}$$

(b) Compute the present age of this universe t_0, as a function of H_0. Is this universe older or younger than a spatially-flat universe entirely filled with non-relativistic matter ($w = 0$) with the same Hubble constant today?

(c) Now imagine that after greatly improving their cosmological measurements, the inhabitants of this universe discover that their universe is in fact *not* spatially flat but has instead a small spatial curvature given by $\Omega_K = -0.01$. Argue that this implies that their universe will eventually stop expanding and start to contract. Assuming the usual normalization of the scale factor such that $a(t_0) = 1$, compute the value of the scale factor at which this turn-around point occurs.

(d) Compute the total age $t_{\rm BC}$ that this universe will have at the time of its "big crunch" when the entire universe has collapsed back to a point (i.e. $a(t_{\rm BC}) = 0$).