PHYS 480/581 - Solutions of Homework 1

Problem 1

Denote t_0 to be the age of the Universe, we have

$$t_0 = \int_0^{t_0} dt = \int_0^{a_0} \frac{da}{Ha}, \tag{1}$$

From the definition of the redshift $1+z \equiv \frac{\lambda_0}{\lambda} = \frac{a_0}{a}$, we have

$$a = \frac{a_0}{1+z} \rightarrow da = \frac{-a_0}{(1+z)^2} dz. \tag{2}$$

So Eq. 1 becomes

$$t_0 = \int_0^{\infty} \frac{dz}{H(1+z)}. \tag{3}$$

Meanwhile, the equation of energy conservation is

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}, \tag{4}$$

The solution of this equation takes the form $\rho \sim a^{-3(1+w)}$. We want to express it in terms of the redshift as

$$\rho = \sum_i \rho_i^{(0)} \left(\frac{a}{a_0}\right)^{-3(1+w_i)} \tag{5}$$

$$= \sum_i \rho_i^{(0)} (1+z)^{3(1+w_i)} \tag{6}$$

$$= \frac{3H_0^2}{8\pi G} \sum_i \Omega_i^{(0)} (1+z)^{3(1+w_i)}, \tag{7}$$

where $\rho_i^{(0)}$ is the energy density of the ith element (dust, radiation, or dark energy) at the present time, $\Omega_i^{(0)} = \frac{\rho_i^{(0)}}{\rho_c^{(0)}}$ is the ratio between energy density of the ith element and critical density at the present time.

The Friedmann equation (for a flat universe) is

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda), \tag{8}$$

where ρ_m, ρ_r, and ρ_Λ are the energy density of matter, radiation, and dark energy, respectively.

Substitute Eq.7 into the Friedmann equation, we get

$$H^2 = H_0^2 \left\{ \Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)} \right\}. \tag{9}$$

Substitute this equation into Eq.3, the age is (we ignore radiation in this problem)

$$t_0 = \int_0^\infty \frac{dz}{H_0(1+z)\sqrt{\Omega_m^{(0)}(1+z)^3 + \Omega_\Lambda^{(0)}}} = 13.4776 \quad Gyr. \tag{10}$$

Problem 2

In the far future, when the universe is completely dominated by dark energy, then the result of problem 1 is

$$t_0 \to \frac{1}{H_0} \ln(1+z) \Big|_0^\infty \to \infty \tag{11}$$

So we see that a far-future alien will determine the age of the universe to be infinite.

The universe thus has the additional temporal symmetry, in the sense that the universe at different times look the same.

Problem 3

a) Without loss of generality, we can say that the photon travels in the radial direction, so that $ds^2 = 0$ implies

$$dr = \pm \frac{dt}{a} \tag{12}$$

We shall choose the plus sign since the comoving distance traveled by photons should increase as time increases. So the comoving distance that a photon will travel from the Big Bang to the epoch of recombination is

$$r = \int_0^{t_{rec}} \frac{dt}{a(t)} \tag{13}$$

It's easy to see that

$$a = \frac{1}{1+z}$$
$$\Rightarrow \frac{da}{dz} = \frac{-1}{(1+z)^2}$$
$$\Rightarrow \frac{da}{dz} = -a^2$$

so that

$$H \equiv \frac{\dot{a}}{a} = \frac{da}{adt}$$
$$\Rightarrow \frac{dt}{a} = \frac{da}{Ha^2} = -\frac{dz}{H(z)}$$

So our integral becomes

$$\begin{aligned}
r &= \int_{z_{rec}}^{\infty} \frac{dz}{H(z)} && (14)\\
&= \int_{z_{rec}}^{\infty} \frac{dz}{H_0\sqrt{\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}}} && (15)\\
&= 8.79125 \times 10^{24} m && (16)\\
&= 284.837 \ \ Mpc && (17)
\end{aligned}$$

where we used the result from problem 1. Note that we brought back the speed of light to get the right unit for distance.

b) From the result of part a, we immediately have the comoving distance traveled by photons from the epoch of recombination to the present time

$$\begin{aligned}
r &= \int_{0}^{z_{rec}} \frac{dz}{H(z)} && (18)\\
&= \int_{0}^{z_{rec}} \frac{dz}{H_0\sqrt{\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}}} && (19)\\
&= 4.34688 \times 10^{26} m && (20)\\
&= 14.084 \times 10^3 \ \ Mpc && (21)
\end{aligned}$$

c) The ratio between the result in part a and part b is 0.0202243, which is 1.15877 degrees. See Fig. 1. Before recombination, photons cannot travel freely, so this angle is in fact the maximum possible angle.

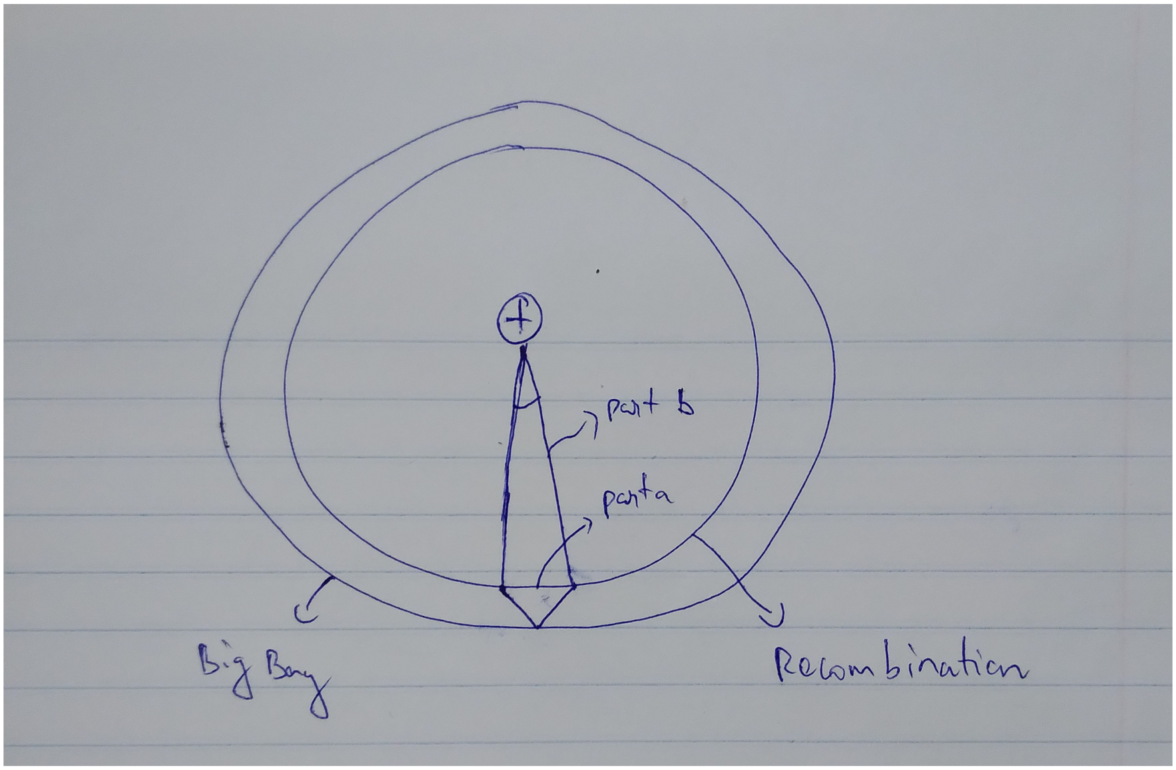

Figure 1: Demonstration of part c, problem 3.

The observed CMB has almost the same temperature from opposite directions in the sky, so this indicates that there exist a horizon problem: the photons can only be causally connected within around 1 degree on the sky, but in reality it seems like photons in all directions are causally connected. This puzzle can be answered by introducing inflation in the early Universe. The basic idea is that the comoving Hubble horizon $(aH)^{-1}$ will decrease exponentially during inflation, so the entire observable universe will be "zoomed out" from a tiny spatial patch and hence photons are easily causally connected in that patch.

Problem 4

a) The conservation of energy equation in an expanding universe is

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a} \tag{22}$$

With the equation of state $\omega = 1$, one can easily solve to get

$$\rho \sim a^{-6} \tag{23}$$

A scalar field which has its kinetic energy dominate its potential energy can realize this scenario (also called "kination era").

b) From the results of problem 1, it's straightforward to see that the age in this case is

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)^4} = \frac{1}{3H_0} \tag{24}$$

For universe that is entirely filled with non-relativistic matter (including normal matter and dark matter), then the age is

$$t_0(matter) = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)^{5/2}} = \frac{2}{3H_0} \tag{25}$$

Compare the above two results, we see that the age of the universe we're considering is less than that of a matter-dominated universe. In fact, one can intuitively expect this without doing any calculations at all by noting that the energy density of a universe containing only non-relativistic matter scales as $\rho \sim a^{-3}$, and hence the matter content dilutes slower (compared to $\rho \sim a^{-6}$) and thus it takes more time to reach the present status of the universe.

c) From the given value $\Omega_K = -0.01 < 0$ and the Friedmann equation of the form $\Omega + \Omega_K = 1$, we can infer that $\Omega > 1$. This means that $\rho > \rho_c$ and hence the universe is closed and will eventually experience a Big Crunch.

To compute the scale factor at the turn-around point, we note that at that moment we must have $\dot{a} = 0$ and hence the Friedmann equation becomes

$$\frac{8\pi G}{3}\frac{\rho_0}{a_{max}^6} = \frac{k}{a_{max}^2}, \tag{26}$$

where ρ_0 is the energy density at the present time. We can express ρ_0 in terms of H_0 by using the Friedmann equation at the present time

$$H_0^2 = \frac{8\pi G}{3}\rho_0 - k \tag{27}$$

So we end up with

$$a_{max} = \left(\frac{8\pi G\rho_0}{3k}\right)^{1/4} = \left(\frac{H_0^2 + k}{k}\right)^{1/4} = \left(\frac{\Omega_k - 1}{\Omega_k}\right)^{1/4} = 3.17015 \tag{28}$$

where we used the definition $\Omega_k \equiv -k/H^2a^2 = -k/H_0^2$. Note that Ω_K is fixed and $a(t_0) = 1$.

d) We note that

$$\frac{k}{a_{max}^2} = \frac{8\pi G}{3}\rho_{min}, \tag{29}$$

so we have

$$\rho = \rho_{min}\left(\frac{a_{max}}{a}\right)^6 = \frac{3k}{8\pi G}\frac{a_{max}^4}{a^6} \tag{30}$$

The Friedmann equation is then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = \frac{k}{a^2}\left(\frac{a_{max}^4}{a^4} - 1\right) \tag{31}$$

$$\Rightarrow t = \int \frac{da}{\sqrt{k}\sqrt{\frac{a_{max}^4}{a^4} - 1}} \tag{32}$$

Set

$$a \equiv a_{max}\left(\frac{1-\cos\theta}{2}\right)^{1/4}$$

$$\Rightarrow \frac{da}{d\theta} = \frac{a_{max}}{2^{1/4}4}(1-\cos\theta)^{-3/4}\sin\theta$$

to get

$$\begin{aligned} t &= \frac{a_{max}}{2^{9/4}\sqrt{k}}\int \frac{\sin\theta d\theta}{(1-\cos\theta)^{3/4}\sqrt{\frac{2}{1-\cos\theta} - 1}} \\ &= \frac{a_{max}}{2^{9/4}\sqrt{k}}\int \frac{\sin\theta}{(1-\cos\theta)^{3/4}}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}d\theta \\ &= \frac{a_{max}}{2^{9/4}\sqrt{k}}\int (1-\cos\theta)^{1/4}d\theta \end{aligned} \tag{33}$$

One can check that the parametric solutions $a(\theta)$ and $t(\theta)$ indeed solve the Friedmann equation. It is apparent from $a(\theta)$ that the Big Bang happens at $\theta = 0$ and the Big Crunch happens at $\theta = 2\pi$. The total age of this universe is then

$$
\begin{aligned}
t_{age} &= \frac{a_{max}}{2^{9/4}\sqrt{k}} \int_0^{2\pi} (1-\cos\theta)^{1/4} d\theta \\
&= \frac{a_{max}}{2^{9/4}\sqrt{k}} 2^{11/4} \times \frac{\Gamma(3/4)^2}{\sqrt{\pi}} \\
&= a_{max}\sqrt{\frac{2}{\pi k}}\Gamma(3/4)^2 \\
&= a_{max}\sqrt{\frac{2}{-\pi H_0^2 \Omega_K}}\Gamma(3/4)^2 \\
&= \frac{37.98}{H_0}
\end{aligned}
\tag{34}
$$