PHYS 480/581 - Solutions of Homework 2

Q1

For nearby sources, we can Taylor expand the scale factor as

$$a(t) \approx a(t_0)[1 + H_0(t - t_0) + ...] \tag{1}$$

From the definition of redshift

$$1 + z = \frac{a(t_0)}{a(t)} \approx \frac{1}{1 + H_0(t - t_0)} \approx 1 - H_0(t - t_0),$$

we obtain

$$z \approx H_0(t_0 - t) = H_0 d \tag{2}$$

We are using the units in which $c = 1$. It does not matter if d is a comoving, angular diameter, or luminosity distance because all these concepts coincide at low reshift. (Recall that $d_A(z) = \chi(z)/(1 + z) \approx \chi(z)$ and $d_L(z) = (1 + z)\chi(z) \approx \chi(z)$ when $z << 1$.)

Q2

It's still reshifted. The reason can be inferred from intuition as follows. Let Δt be the time interval between the moment light is emitted and the moment when the universe starts to recollapse. If the observer in another galaxy receives the signal at the time larger than Δt after the turn-over moment, then the light received is blueshifted. Otherwise it is redshifted, because there is not sufficient expansion to squeeze the wavelength with the amount that it was stretched (temporal symmetry of a big bang-big crunch is implicitly used). In our question, the observer receives light *just after* the turn-over point, so she still sees the light redshifted.

Q3

The comoving distance is

$$\chi(z) = \int_0^z \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H_0\sqrt{0.3(1 + z')^3 + 0.7}} \tag{3}$$

So that

$$\frac{\theta(z)}{L} = \frac{1}{d_A(z)} = \frac{1+z}{\chi(z)} \tag{4}$$

This function is plotted in Fig. 1.

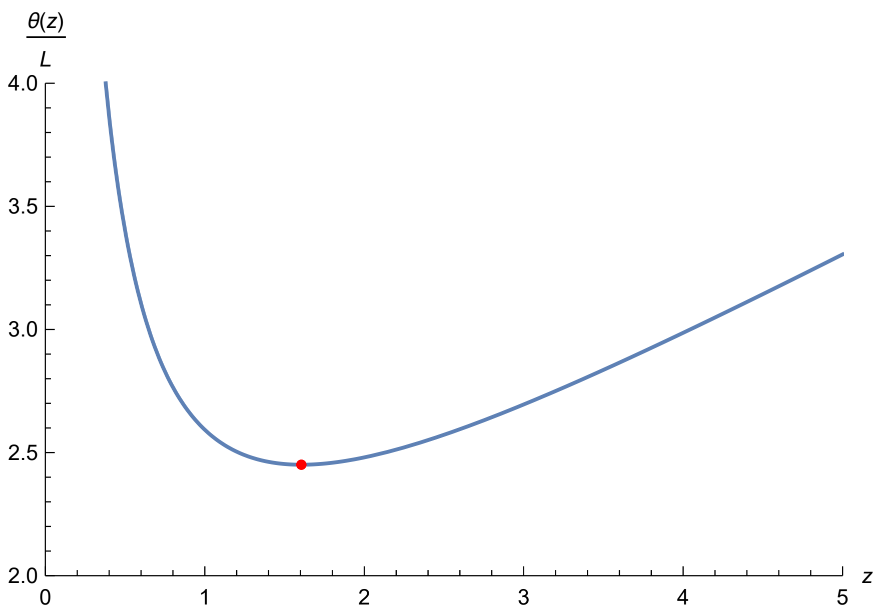

Figure 1: $\theta(z)/L$ (in units of H_0) as a function of redshift. The red dot corresponds to the minimum.

From the figure, we can find

$$\left(\frac{\theta(z)}{L}\right)_{min} = 2.451 \times H_0, \quad at \;\; z = 1.605 \tag{5}$$

For $L = 10$ kpc, we get

$$\theta\Bigg|_{z=1.605} = 2.451 \times H_0 \times L = 5.719 \times 10^{-6} \;\; rad = 1.18 \;\; arcseconds. \tag{6}$$

If this galaxy is at $z = 10$, we get

$$\theta\Bigg|_{z=10} = 11.6 \times 10^{-6} \;\; rad = 2.39 \;\; arcseconds. \tag{7}$$

Q4

a)

As the question suggested, we can simply get from the results we had in homework No. 1 that the age of the universe at redshift z is

$$t(z) = \int_z^\infty \frac{dz'}{H_0(1+z')\sqrt{\Omega_m(1+z')^3 + \Omega_r(1+z')^4 + \Omega_\Lambda}} \tag{8}$$

See Fig. 2

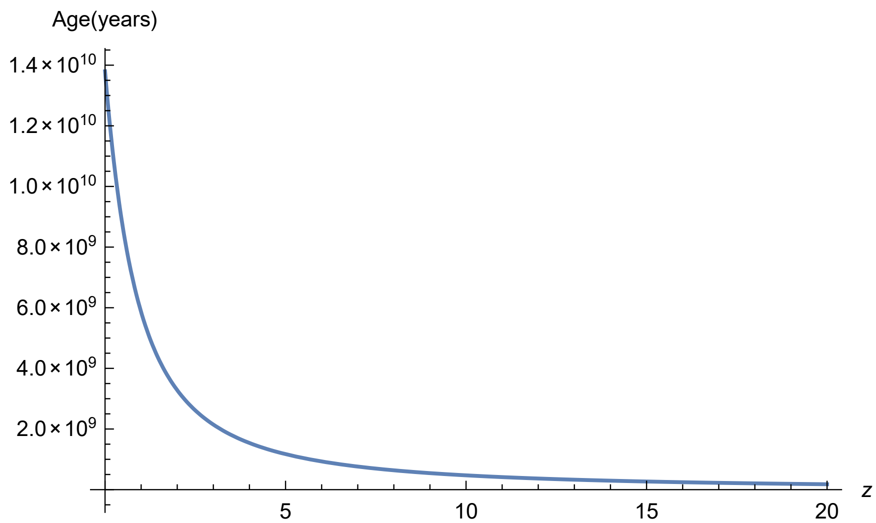

Figure 2: The age of the universe as a function of redshift.

b)

The fraction of the current age of the universe has elapsed by $z = 2$ is

$$\frac{t(z=2)}{t(z=0)} \approx 23.77\% \tag{9}$$

The other case is

$$\frac{t(z=10)}{t(z=0)} \approx 3.42\% \tag{10}$$

c)

At recombination, the age of the universe is

$$t(z = 1090) = 372310 \;\; years \tag{11}$$

Q5

a)

The data is plotted in Fig. 3

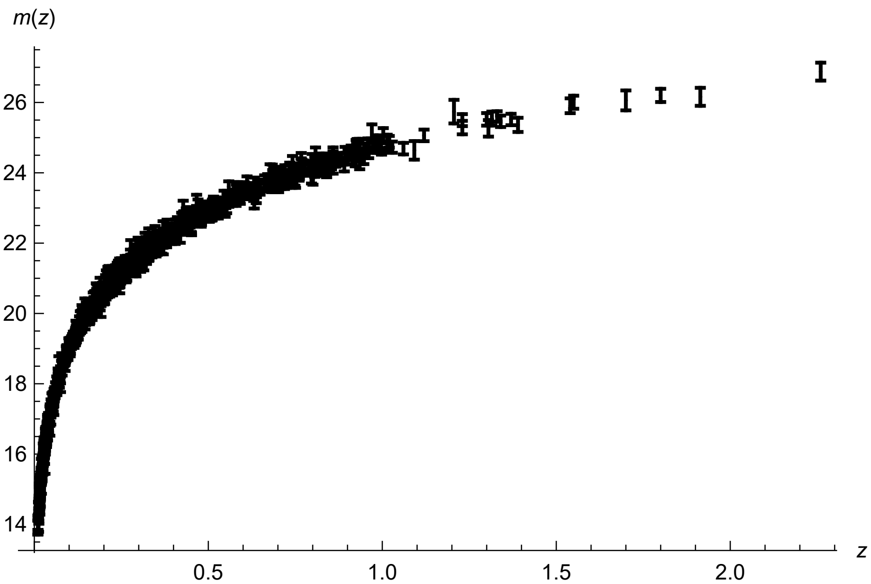

Figure 3: Observations of apparent magnitude as a function of redshift.

b)

The luminosity distance at redshift z is

$$d_L(z) = (1+z)^2 d_A(z) = (1+z)\chi(z) = \frac{1+z}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}} \quad (12)$$

Just throw in some numbers and use the equations given in the question, we get Fig. 4.

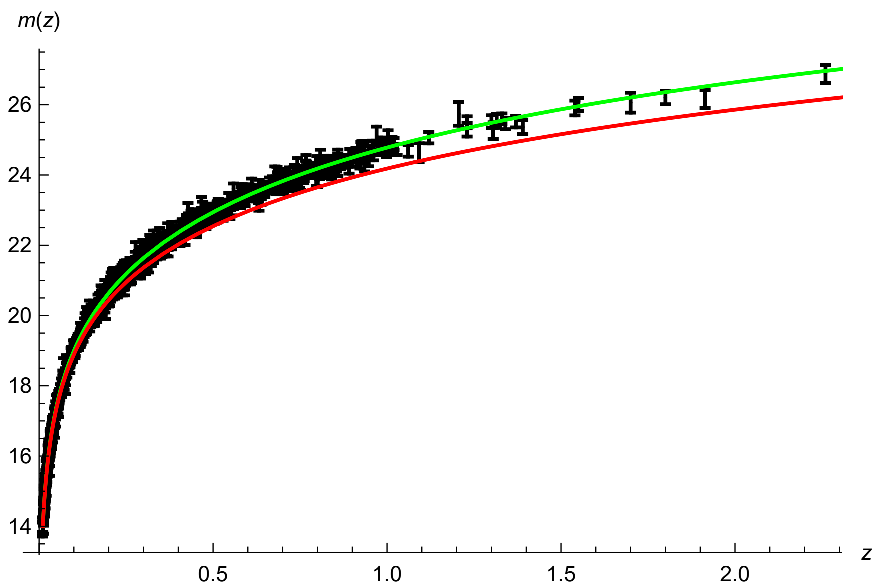

Figure 4: Comparisons between theoretical predictions and observations. The green solid curve corresponds to the model with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. The red solid curve corresponds to the model with $\Omega_m = 1$.

c)

We see that the model with a cosmological constant better fits the data. If we change M to the value -18.9 (suggested by the instructor), the model with $\Omega_m = 1$ can fit the data at high redshifts but gives a poor fit at small reshifts. See Fig. 5. So changing M cannot really improve the fit.

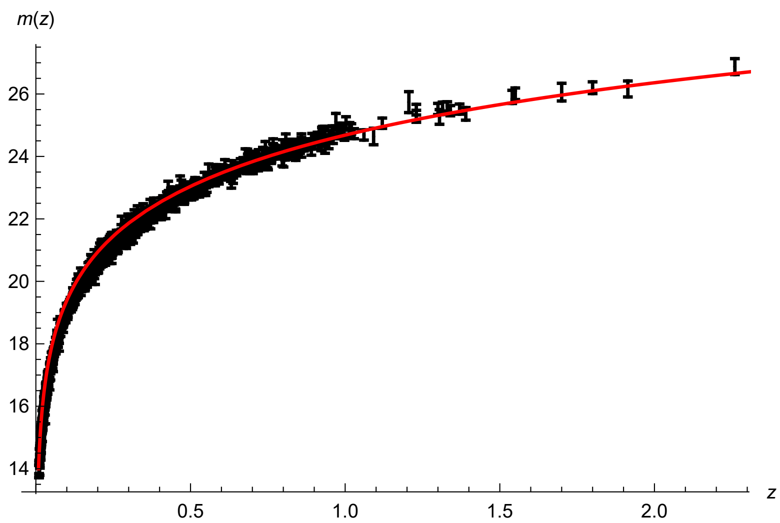

Figure 5: Comparison between observations and the theory prediction of model $\Omega_m = 1$ with $M = -18.9$ (red solid curve.)

Q6

a)

Recall that

$$\frac{\Omega_\Lambda}{\Omega_m} \sim a^3 \tag{13}$$

(Because ρ_Λ is constant and $\rho_m \sim a^{-3}$). We can scale this as

$$\frac{\Omega_\Lambda}{\Omega_m} = \frac{\Omega_\Lambda^{eq}}{\Omega_m^{eq}} \frac{a^3}{a_{eq}^3} = \frac{a^3}{a_{eq}^3} \tag{14}$$

where the notation "eq" denotes the value of quantities at matter-dark energy equality. So, for the present time we get

$$\frac{\Omega_\Lambda^0}{\Omega_m^0} = \frac{1}{a_{eq}^3} = (1 + z_{eq})^3 \tag{15}$$

$$\Rightarrow z_{eq} = \left(\frac{\Omega_\Lambda^0}{\Omega_m^0}\right)^{1/3} - 1 \tag{16}$$

Just plug in the given values to get $z_{eq} = 0.304$. It is interesting to note that this is actually *not* the time when the universe began to accelerate. By doing some simple calculations based on the acceleration equation, one can indeed show that the moment when the universe began to accelerate again (after its first time during inflation) is $z_{accelerate} = (2\Omega_\Lambda^0/\Omega_m^0)^{1/3} - 1$. This means that the universe already started to accelerate when it is still matter dominated. It took some time for dark energy to take over and dominate the energy content.

The time interval that has been elapsed between that epoch and today is (use results from Q4)

$$t(z=0) - t(z=0.304) \approx 3.6 \ Gyrs \tag{17}$$

This is just 1 billion years smaller than the age of the Solar system, so it is still acceptable in cosmological terms to say that there is a "coincidence" problem.

b)

Recall that

$$\frac{\Omega_m}{\Omega_r} \sim a \tag{18}$$

because $\rho_m \sim a^{-3}$ and $\rho_r \sim a^{-4}$. We can scale this as

$$\frac{\Omega_m}{\Omega_r} = \frac{\Omega_m^{eq}}{\Omega_r^{eq}} \frac{a}{a_{eq}} = \frac{a}{a_{eq}} \tag{19}$$

So at the present time we get

$$\frac{\Omega_m^0}{\Omega_r^0} = \frac{1}{a_{eq}} = 1 + z_{eq} \tag{20}$$

where "eq" now means the matter-radiation equality moment. So that

$$z_{eq} = \frac{\Omega_m^0}{\Omega_r^0} - 1 \tag{21}$$

We can use the same value of Ω_m^0 that is given in Q4, but not Ω_r^0 since now we have two extra flavors of massless neutrinos.

According to the Standard Model of particle physics, neutrinos are massless. The total number of relativistic degrees of freedom of neutrinos is 10 (because massless neutrinos have 1 degree of freedom and we have 5 neutrinos and their antiparticle partners), while for photon it is still 2. The energy density of radiation is then

$$\rho_r^0 = \frac{\pi^2}{30}\left(2T_\gamma^4 + \frac{7}{8} \times 10 \times T_\nu^4\right) \tag{22}$$

$$= \frac{\pi^2}{30}\left(2T_\gamma^4 + \frac{7}{8} \times 10 \times \left(\frac{4}{11}\right)^{4/3} T_\gamma^4\right) \tag{23}$$

$$= \frac{\pi^2}{30}\left(2 + \frac{35}{4}\left(\frac{4}{11}\right)^{4/3}\right) T_\gamma^4 \tag{24}$$

$$= \frac{\pi^2}{30}\left(2 + \frac{35}{4}\left(\frac{4}{11}\right)^{4/3}\right) T_\gamma^4 \frac{k_B^4}{\hbar^3 c^3} \tag{25}$$

where we used $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$ and in the last line we threw back some constants to get the right units for energy density. The temperature of CMB today is $T_\gamma = 2.725$ K. The critical energy density at the moment is

$$\rho_c^0 = \frac{3H_0^2}{8\pi G} = \frac{3H_0^2}{8\pi G} c^2 \tag{26}$$

So we have, in this hypothetical model, that

$$\Omega_r^0 = \rho_r^0/\rho_c^0 = 1.16 \times 10^{-4} \tag{27}$$

As a check, one can just assume the conventional three neutrino flavors and one can indeed reproduce the value 9.1×10^{-5} given in the Q4. The redshift at matter-radiation equality in this hypothetical model is then

$$z_{eq} = \frac{0.311}{1.16 \times 10^{-4}} - 1 \approx 2680. \tag{28}$$

As a side note, the standard case with 3 neutrino flavors (and hence $\Omega_r = 9.1 \times 10^{-5}$) gives the famous value $z_{eq} \sim 3400$.