PHYS 480/581 - Solutions of Homework 3

Q1

a)

By integrating the angular parts of the integral and changing variables, we find

$$n = \frac{g}{2\pi^2}\int_0^\infty \frac{p^2}{e^{\sqrt{p^2+m^2}/T}+1}dp \tag{1}$$

$$= \frac{g}{2\pi^2}\int_0^\infty \frac{x^2T^3}{e^{\sqrt{x^2+y^2}}+1}dx \tag{2}$$

$$\Rightarrow \frac{n(y)}{T^3} = \frac{g}{2\pi^2}\int_0^\infty \frac{x^2}{e^{\sqrt{x^2+y^2}}+1}dx \tag{3}$$

b)

The function $n(y)/T$ for $g = 2$ is plotted in Fig. 1.

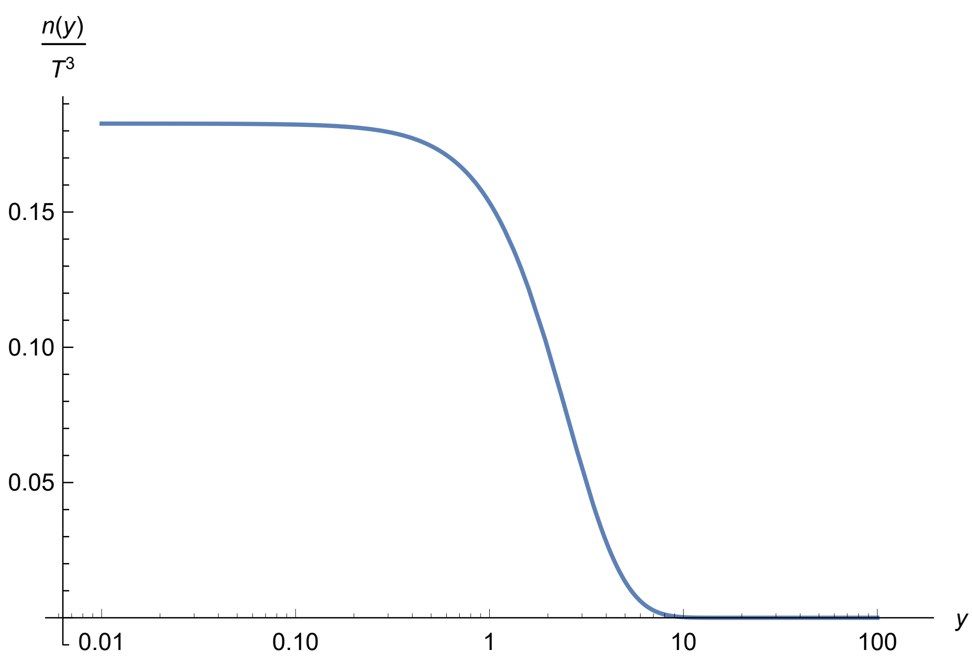

Figure 1: $n(y)/T^3$ as a function of y for $g = 2$.

c)

Recall that in the relativistic limit ($y << 1$) we have

$$\frac{n(y)}{T^3} = \frac{\xi(3)}{\pi^2} g \frac{3}{4} = \frac{3\xi(3)}{2\pi^2} \tag{4}$$

where $\xi(3) \approx 1.202$.

In the non-relativistic limit ($y >> 1$) we have

$$\frac{n(y)}{T^3} = \frac{g}{(2\pi)^{3/2}} \left(\frac{m}{T}\right)^{3/2} e^{-m/T} = \frac{2}{(2\pi)^{3/2}} y^{3/2} e^{-y} \tag{5}$$

So, we add these two curves in Fig. 2.

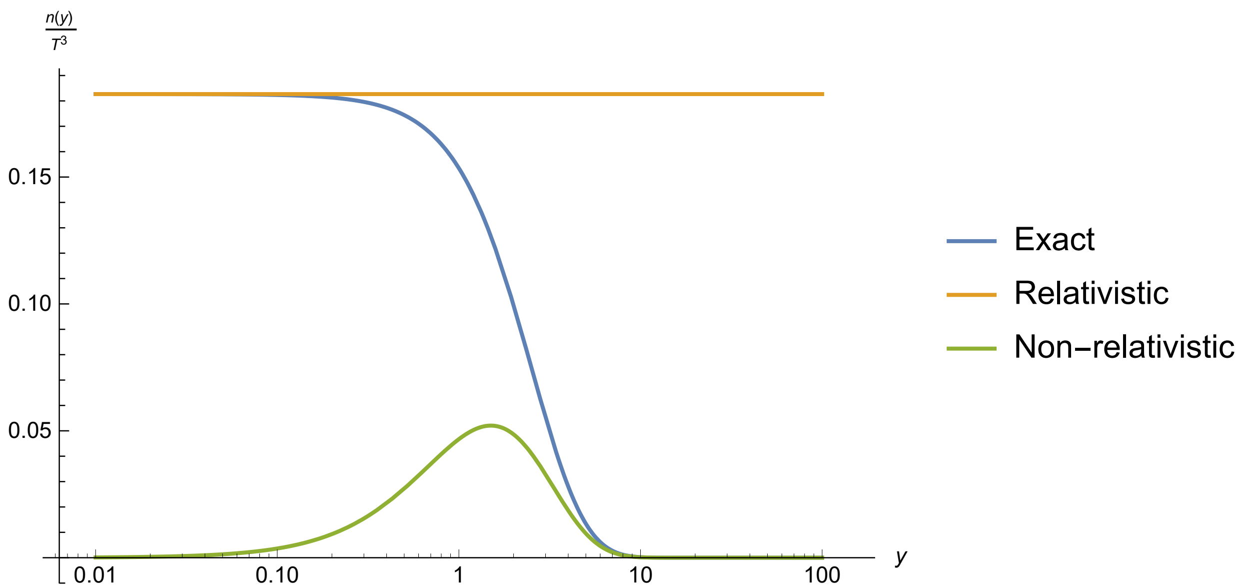

Figure 2: Comparison between the exact number density with the relativistic and non-relativistic limits.

We see that the results in the relativistic limit and non-relativistic limit match the exact result in the appropriate regimes.

Q2

a)

The number density of fermions is

$$n = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left(\frac{\sqrt{p^2+m^2}-\mu}{T}\right)+1} \tag{6}$$

In the relativistic limit $T >> m$, we have

$$n - \bar{n} = \frac{g}{2\pi^2} \int_0^\infty dp p^2 \left(\frac{1}{e^{(p-\mu)/T}+1} - \frac{1}{e^{(p+\mu)/T}+1} \right), \tag{7}$$

where we performed the angular part of the integral with $\int d\Omega = 4\pi$ and used the fact that $\mu_{\bar{f}} = -\mu_f$.

We will use several mathematical tricks to go to the final result. First, set $x = p/T$ and $z = \mu/T$. So, we get

$$n - \bar{n} = \frac{gT^3}{2\pi^2} \int_0^\infty dx x^2 \left(\frac{1}{e^{x-z}+1} - \frac{1}{e^{x+z}+1} \right) \tag{8}$$

Next, set $u = x - z$ for the first integral and $u = x + z$ for the second integral, we get

$$\begin{aligned}
n - \bar{n} &= \frac{gT^3}{2\pi^2} \left[\int_{-z}^\infty du \frac{(u+z)^2}{e^u+1} - \int_z^\infty du \frac{(u-z)^2}{e^u+1} \right] \\
&= \frac{gT^3}{2\pi^2} \left[\int_{-z}^0 du \frac{(u+z)^2}{e^u+1} + \int_0^\infty du \frac{(u+z)^2}{e^u+1} - \int_z^0 du \frac{(u-z)^2}{e^u+1} - \int_0^\infty du \frac{(u-z)^2}{e^u+1} \right] \\
&= \frac{gT^3}{2\pi^2} \left[-\int_z^0 du \frac{(-u+z)^2}{e^{-u}+1} - \int_z^0 du \frac{(u-z)^2}{e^u+1} + \int_0^\infty du \frac{4uz}{e^u+1} \right] \\
&= \frac{gT^3}{2\pi^2} \left[\int_0^z du (u-z)^2 \left(\frac{1}{e^{-u}+1} + \frac{1}{e^u+1} \right) + \int_0^\infty du \frac{4uz}{e^u+1} \right] \\
&= \frac{gT^3}{2\pi^2} \left[\int_0^z du (u^2 - 2uz + z^2) + 4z \int_0^\infty du \frac{u}{e^u+1} \right] \\
&= \frac{gT^3}{2\pi^2} \left[\frac{z^3}{3} - z^3 + z^3 + 4z \frac{\pi^2}{12} \right] \\
&= \frac{gT^3}{6\pi^2} \left[z^3 + \pi^2 z \right] \\
&= \frac{gT^3}{6\pi^2} \left[\left(\frac{\mu}{T}\right)^3 + \pi^2 \left(\frac{\mu}{T}\right) \right].
\end{aligned}$$

In the first line, we just replaced the dummy variable x by u. In the second line, we just separate two integrals to four integrals. In the third line, we rearranged some terms and switched the variable $u \to -u$ in the first integral. In the fourth line, we just rearranged some terms. In the fifth line, we realized that $\frac{1}{e^{-u}+1} + \frac{1}{e^{u}+1} = 1$. The rest is just doing some integrals and using the fact that

$$\int_0^\infty du \frac{u}{e^u + 1} = \frac{\pi^2}{12}.$$

b)

When $T >> 1$ MeV, the number densities of proton and neutron are equal because the exponential factor $n_n/n_p \sim \exp(-(m_n - m_p)/T) \sim \exp(-1.3MeV/T)$ approaches 1. (The proportional constant is $(m_n/m_p)^{3/2}$ and can be neglected as it is very close to 1, too) . Baryons are mostly protons and neutrons. So, we have

$$n_p = n_n = \frac{n_b}{2} = \frac{1}{2}\frac{n_b}{n_\gamma} n_\gamma = \frac{1}{2}\eta_b n_\gamma = \frac{\eta_b}{2} \times \frac{2\xi(3)}{\pi^2} T^3 = \frac{\xi(3)\eta_b}{\pi^2} T^3 \tag{9}$$

where we used the fact that the number density of photons is $n_\gamma = 2\xi(3)T^3/\pi^2$.

Next, we note that the chemical potential of electron is $\mu_e \approx m_e$, so the ratio μ_e/T is very small when $T >> 1$ MeV. So, the result in part a reduces to

$$n_p \approx \frac{T^3}{3\pi^2}\pi^2 \left(\frac{\mu_e}{T}\right) = \frac{T^3}{3}\left(\frac{\mu_e}{T}\right) \tag{10}$$

Note that $g = 2$ for electron. Compare equations 9 and 10, we get

$$\frac{\mu_e}{T} \approx \frac{3\eta_b\xi(3)}{\pi^2}. \tag{11}$$

Q3

a)

Recall that the number density of relativistic particles is

$$n = \frac{\xi(3)}{\pi^2} g T^3 \begin{cases} 1 : bosons \\ 3/4 : fermions \end{cases} \tag{12}$$

For one generation of neutrinos, we have $g_\nu = 2$. For photons, we have $g_\gamma = 2$. So, we get

$$\frac{n_\nu}{n_\gamma} = \frac{3 g_\nu T_\nu^3}{4 g_\gamma T_\gamma^3} = \frac{3}{4}\frac{T_\nu^3}{T_\gamma^3} = \frac{3}{4} \times \frac{4}{11} = \frac{3}{11} \tag{13}$$

b)

We recall that the number density of CMB photons today is

$$n_{\gamma,0} = 410 \ \ cm^{-3} \tag{14}$$

And the critical density today is

$$\rho_{c,0} = 1.05 \times 10^4 \ \frac{h^2 eV}{cm^3} \tag{15}$$

For non-relativistic massive neutrinos, we have $\rho_\nu = m_\nu n_\nu$. The result in part a holds for each generation of neutrinos, so we get

$$\Omega_{\nu,0} h^2 = \frac{\rho_{\nu,0}}{\rho_{c,0}} h^2 = \frac{n_{\nu,0} \sum_i m_{\nu,i}}{\rho_{c,0}} h^2 = \frac{3}{11} \times \frac{410}{1.05 \times 10^4 eV} \sum_i m_{\nu,i} \approx \frac{\sum_i m_{\nu,i}}{94 eV}. \tag{16}$$

Q4

a)

Recall that the energy density of relativistic particles is

$$\rho = \frac{\pi^2}{30} g T^4 \begin{cases} 1 : bosons \\ 7/8 : fermions \end{cases} \tag{17}$$

So, we get

$$\frac{\rho_\nu}{\rho_\gamma} = \frac{7}{8} \times \frac{3 \times 2}{2} \left(\frac{T_\nu}{T_\gamma}\right)^4 = \frac{21}{8} \left(\frac{4}{11}\right)^{4/3} \tag{18}$$

b)

Because the masses of neutrinos are small, it is a reasonable assumption that all neutrinos have the same temperature. The energy density of neutrinos is then

$$\rho_\nu(z) = \frac{\pi^2}{30} \times 2 \times \frac{7}{8} T_\nu^4 + 2\int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2+m_2^2}}{e^{p/T_\nu}+1} + 2\int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2+m_3^2}}{e^{p/T_\nu}+1}$$
$$= \frac{\pi^2}{30} \times 2 \times \frac{7}{8}(1+z)^4 T_{\nu,0}^4 + \frac{1}{\pi^2}\left(\int_0^\infty \frac{p^2\sqrt{p^2+m_2^2}dp}{\exp\left(\frac{p}{(1+z)T_{\nu,0}}\right)+1} + \int_0^\infty \frac{p^2\sqrt{p^2+m_3^2}dp}{\exp\left(\frac{p}{(1+z)T_{\nu,0}}\right)+1}\right),$$

where $T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,0}$. The energy density of photons as a function of redshift is

$$\rho_\gamma(z) = \frac{\pi^2}{30} \times 2 \times T_\gamma^4 = \frac{\pi^2}{30} \times 2(1+z)^4 T_{\gamma,0}^4 \tag{19}$$

Recall that $T_{\gamma,0} = 2.725K = 2.35 \times 10^{-4}$ eV. So, we have Fig. 3.

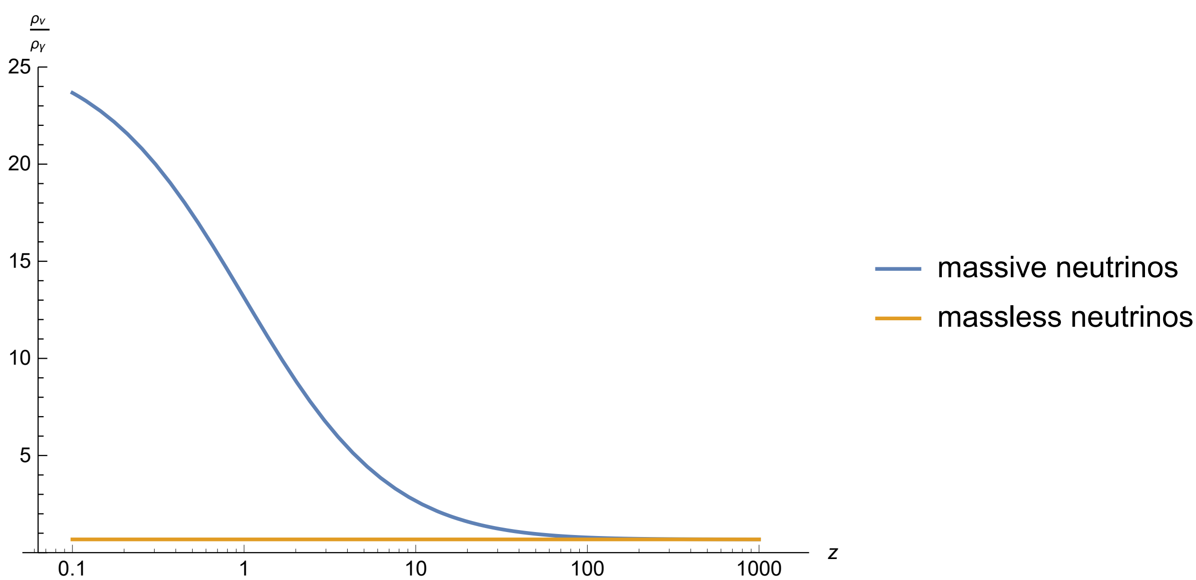

Figure 3: $\rho_\nu(z)/\rho_\gamma(z)$ for a realistic model (blue curve) and an ideal model (orange curve) as functions of redshift.

Two remarks:

- The curve of massive neutrinos asymptotically approaches the curve of massless neutrinos at high redshifts. This is expected because the temperature is high at high redshifts so that particles are relativistic.

- At low redshifts, the energy density of massive neutrinos is larger than that of massless neutrinos. This is because the energy density of non-relativistic particles dilutes slower than relativistic ones.