PHYS 480/581
Cosmology

Homework Assignment 4
Due date: Wednesday November 9 2022, in class

Question 1 (6 points).

Using the expression for the pressure P in terms of the particle distribution function $f(p)$

$$P = g \int \frac{d^3p}{(2\pi)^3} f(p) \frac{p^2}{3E}, \tag{1}$$

to show that, in thermal equilibrium with vanishing chemical potential, we have

$$\frac{\partial P}{\partial T} = \frac{\rho + P}{T}. \tag{2}$$

To do so, first argue that in thermal equilibrium, the particle distribution function only depends on E/T. Then use the chain rule to relate $\partial f/\partial T$ to $\partial f/\partial p$. Finally, use integration by parts to obtain Eq. (2). You should never have to explicitly write down $f(p)$ in terms of its Bose-Einstein or Fermi-Dirac form.

Question 2 (6 points).

In class, we argued that the neutron freeze-out occurs at a temperature of $T_f \simeq 0.8$ MeV. This temperature can be estimated by comparing the weak interaction rate of the neutrons with the Hubble rate at that epoch. The interaction rate for the key reactions $p + \bar{\nu}_{\rm e} \leftrightarrow n + e^+$ and $p + e^- \leftrightarrow n + \nu_{\rm e}$ is given by

$$\Gamma_{\rm W}(x) = \left(\frac{255}{\tau_n}\right) \frac{12 + 6x + x^2}{x^5}, \tag{3}$$

where $\tau_n = 886.7$ sec is the neutron lifetime, and $x = Q/T$, with $Q \equiv m_n - m_p = 1.2933$ MeV. On the other hand, the Hubble expansion rate can be gotten from the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho_{\rm rad}, \qquad \text{with} \qquad \rho_{\rm rad} = \frac{\pi^2}{30} g_*(T) T^4. \tag{4}$$

Heuristically, neutron freeze-out will occur when $\Gamma_{\rm W} \sim H$. A more precise value for $T_{\rm f}$ can be obtained by solving

$$\Gamma_{\rm W}(T_f) = \frac{3}{2} H(T_f). \tag{5}$$

Using the expressions given above, show that $T_f \simeq 0.8$ MeV. Perhaps, the easiest way to do that is to plot both $\Gamma_{\rm W}(T)$ and $H(T)$ and determine where they intersect. Be mindful of the units to make sure you are comparing the two rates in the same unit system. What value of $g_*(T)$ should you use in the above?

Question 3 (6 points).

In class, we mentioned several time that the age of the Universe was about 1 second when neutron froze out at $T_f = 0.8$ MeV. Let us derive this result. First, remember that the age of the Universe at scale factor a is given by

$$t(a) = \int_0^a \frac{da'}{a'H(a')}. \quad (6)$$

The issue is that we don't know H as a function of a at early times, but rather as a function of temperature

$$H(T) = \sqrt{\frac{8\pi G}{3}\rho_{\rm rad}(T)}, \qquad \text{with} \qquad \rho_{\rm rad} = \frac{\pi^2}{30} g_*(T) T^4. \quad (7)$$

To make matter worse, T does not scale as $1/a$ when g_* (or g_{*S}) is changing. However, we can derive an approximate expression that is pretty accurate by taking $g_*(T)$ to be constant and $T \propto 1/a$.

(a) Show that if $T \propto 1/a$, we have

$$\frac{dT}{T} = -\frac{da}{a}. \quad (8)$$

(b) Using the above and assuming $g_*(T)$ to be constant, show that the age of the Universe at temperature T was

$$\frac{t}{\rm sec} \simeq \frac{2.42}{\sqrt{g_*}} \left(\frac{T}{\rm MeV}\right)^{-2}. \quad (9)$$

(c) Using the appropriate value of g_* for $T \sim 1$ MeV, show that the age of the Universe at $T_f = 0.8$ MeV was $t \simeq 1.15$ sec.

Question 4 (3 points).

Use the Saha equation for the free electron fraction X_e,

$$\left(\frac{1 - X_e}{X_e^2}\right)_{\rm eq} = \frac{2\zeta(3)}{\pi^2}\eta_{\rm b}\left(\frac{2\pi T}{m_{\rm e}}\right)^{3/2} e^{B_{\rm H}/T}, \quad (10)$$

to show that the temperature of the Universe when 90% of the electrons have combined with protons to form neutral atoms (i.e. $X_e = 0.1$) is $T_{\rm rec} \simeq 0.3$ eV. Why is $T_{\rm rec} \ll B_{\rm H} = 13.6$ eV?