PHYS 480/581 - Solutions of Homework 4

Q1

The Fermi-Dirac/Bose-Einstein distributions for fermions/bosons with zero chemical potential is

$$f = \frac{1}{e^{E/T} \pm 1}. \tag{1}$$

This means that $f(E/T)$ is a function of only E/T. From this fact, we have

$$\frac{\partial f}{\partial T} = \frac{\partial f}{\partial E}\frac{\partial E}{\partial(E/T)}\frac{\partial(E/T)}{\partial T} = -\frac{E}{T}\frac{\partial f}{\partial E} \tag{2}$$

For the time being, let's just forget about the angular part and the proportional constant. We focus on the integral over p. The pressure is

$$P \propto \int dp \frac{p^4}{3E} f(p) \tag{3}$$

$$\begin{aligned}
\Rightarrow \frac{\partial P}{\partial T} &\propto \int_0^\infty dp \frac{p^4}{3E}\frac{\partial f}{\partial T} \\
&= -\int_0^\infty dp \frac{p^4}{3T}\frac{\partial f}{\partial E} \\
&= -\int_m^\infty \frac{EdE}{p}\frac{p^4}{3T}\frac{\partial f}{\partial E} \\
&= -\frac{1}{3T}\int_m^\infty dE E(E^2 - m^2)^{3/2}\frac{\partial f}{\partial E} \\
&= \frac{1}{3T}\int_m^\infty dE \left[(E^2 - m^2)^{3/2} + 3E^2(E^2 - m^2)^{1/2}\right] f \\
&= \frac{1}{3T}\int_0^\infty \frac{pdp}{E} \left[(E^2 - m^2)^{3/2} + 3E^2(E^2 - m^2)^{1/2}\right] f \\
&= \frac{1}{3T}\int_0^\infty dp \left[\frac{p^4}{E} + 3Ep^2\right] f \\
&= \frac{1}{T}\int_0^\infty p^2 dp \left[\frac{p^2}{3E} + E\right] f.
\end{aligned}$$

In the above calculations, we used Eq. 2 and this fact: $E = \sqrt{p^2 + m^2} \Rightarrow dE/dp = p/E$; this is used to convert the integral over p to the integral over

E and vice versa. Also note that $f(E \to \infty) = 0$ when doing the integration by parts in the fifth line.

We can now recover the angular part and the proportional constant to get

$$\frac{\partial P}{\partial T} = \frac{1}{T}\frac{g}{(2\pi)^3}\int d^3p\left[\frac{p^2}{3E} + E\right] f = \frac{P+\rho}{T}. \tag{4}$$

Note that $P = \frac{g}{(2\pi)^3}\int d^3p\frac{p^2}{3E}f(p)$ and $\rho = \frac{g}{(2\pi)^3}\int d^3pEf(p)$.

Q2

All we need to do is converting the units of Hubble rate to second:

$$H^2 = \frac{8\pi G}{3}\rho_{rad} = \frac{4\pi^3 g_*(T)}{45}\frac{G}{\hbar^3c^5}\left(\frac{T}{MeV}1.6\times 10^{-13}J\right)^4 (sec^{-2}) \tag{5}$$

At neutron freeze-out, we shall have $g_* = 10.75$. The weak decay rate was given as

$$\Gamma_W = \frac{255}{\tau_n}\frac{12+6x+x^2}{x^5}, \tag{6}$$

where the neutron lifetime is $\tau_n = 886.7$ sec and $x = Q/T$ with $Q \equiv m_n - m_p = 1.2933$ MeV. The two rates Γ_W and $3H/2$ are plotted in Fig. 1. The two curves intersect at $T_f \simeq 0.8$ MeV, as expected.

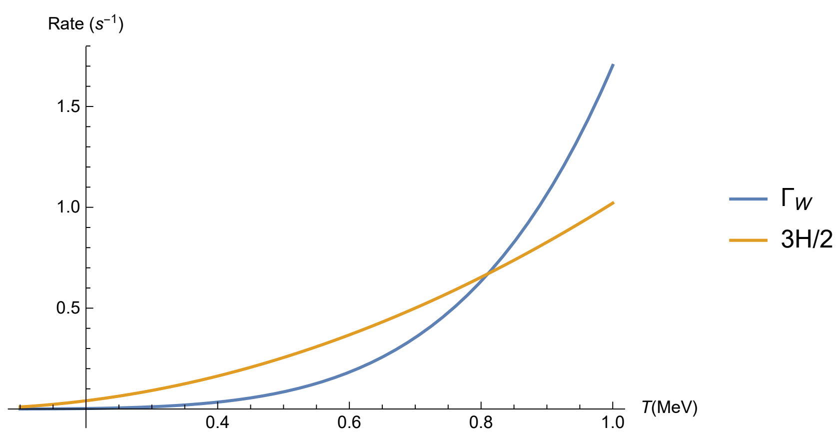

Figure 1: Blue line: the weak decay rate Γ_W. Orange line: 3H/2.

Q3

a)

We have

$$T \propto \frac{1}{a} \Rightarrow \frac{dT}{da} \propto -\frac{1}{a^2}, \tag{7}$$

so that

$$\frac{dT}{Tda} = -\frac{1}{a} \Rightarrow \frac{dT}{T} = -\frac{da}{a} \tag{8}$$

b)

Using the result of part (a), we get

$$\begin{aligned}
t &= \int_0^a \frac{da'}{a'\sqrt{\frac{8\pi G}{3}\frac{\pi^2}{30}g_* T^4}} \\
&= \sqrt{\frac{45}{4\pi^3 G g_*}} \int_0^a \frac{da'}{a' T^2} \\
&= \sqrt{\frac{45}{4\pi^3 G g_*}} \int_T^\infty \frac{dT'}{T'^3} \\
&= \frac{1}{2}\sqrt{\frac{45}{4\pi^3 G g_*}} T^{-2} \\
&= \frac{1}{2}\sqrt{\frac{45}{4\pi^3 G g_*}} \sqrt{\frac{\hbar^3 c^5}{(1.6\times 10^{-13} J)^4}} \left(\frac{T}{MeV}\right)^{-2} sec \\
&\simeq \frac{2.42}{\sqrt{g_*}} \left(\frac{T}{MeV}\right)^{-2} sec
\end{aligned}$$

c)

We mentioned some times ago that $g_*(T \sim 1 MeV) = 10.75$, so we get

$$t(T_f \simeq 0.8\ MeV) \simeq \frac{2.42}{\sqrt{10.75}} \left(\frac{0.8\ MeV}{MeV}\right)^{-2} sec \simeq 1.15\ sec. \tag{9}$$

Q4

One can use the iteration method discussed in class (see Appendix), but here we shall use a plotting method because it's faster. We plot both sides of the Saha equation in Fig. 2. Note that $X_e = 0.1$, $\xi(3) = 1.202$, $\eta_b = 6 \times 10^{-10}$,

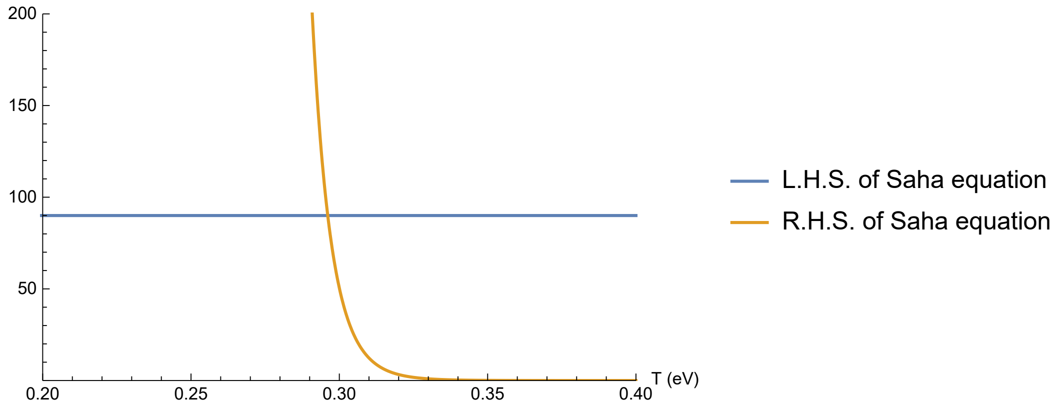

Figure 2: Blue line: the left-hand side of Saha equation. Orange line: the right-hand side of Saha equation.

$B_H = 13.6$ eV, and $m_e = 0.511 \times 10^6$ eV. The two curves intersect at $T_{rec} \simeq 0.3$ eV.

The baryon-to-photon ratio η_b is very small, which means that there are many photons for each hydrogen atom. This implies that even if we have $T < B_H$, there is a sufficient number of high-energy photons, corresponding to the high-energy tail of the photon distribution, that can ionize the hydrogen atoms. So, we have to wait until $T_{rec} << B_H$ so that the exponential term can kill η_b.

Appendix

Here we provide an alternative solution of Q4. The Saha equation is

$$\left(\frac{1-X_e}{X_e^2}\right)_{eq} = \frac{2\xi(3)}{\pi^2}\eta_b \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T}. \tag{10}$$

Taking the logarithm of both sides, we get

$$\ln\left(\frac{1-X_e}{X_e^2}\right) = \ln\left(\frac{2\xi(3)\eta_b}{\pi^2}\right) + \frac{3}{2}\ln\left(\frac{2\pi T}{m_e}\right) + \frac{B_H}{T}. \tag{11}$$

The first iteration solution is

$$T_{(1)} = B_H / \ln\left(\frac{1 - X_e}{X_e^2}\frac{\pi^2}{2\xi(3)\eta_b}\right) \simeq 0.5 \;\; eV. \tag{12}$$

Note that $X_e = 0.1$, $\xi(3) = 1.202$, $\eta_b = 6 \times 10^{-10}$, $B_H = 13.6$ eV, and $m_e = 0.511 \times 10^6$ eV. Substituting this back into the logarithm term of Eq. 11, we get the second iteration solution

$$T_{(2)} = B_H / \ln\left(\frac{1 - X_e}{X_e^2}\frac{\pi^2}{2\xi(3)\eta_b}\left(\frac{m_e}{2\pi T_{(1)}}\right)^{3/2}\right) \simeq 0.3 \;\; eV. \tag{13}$$