PHYS 480/581
Cosmology

Homework Assignment 5

Due date: Wednesday November 23 2022, in class

Question 1 (3 points).

Even in the absence of recombination (that is, $X_e = n_\mathrm{e}/(n_\mathrm{p} + n_\mathrm{H}) = 1$ at all times), the photons populating our Universe would eventually decouple from the baryons anyway due the volumetric dilution from the expansion. Estimate the temperature and redshift at which photon decoupling would occur in a Universe that is always ionized. Use $H_0 = 2.133h \times 10^{-33}$ eV, $h = 0.674$, $\Omega_\mathrm{m} = 0.315$, $\eta_\mathrm{b} = 6.1 \times 10^{-10}$, and $\sigma_\mathrm{T} = 1.71 \times 10^{-3}$ MeV^{-2}.

Question 2 (10 points).

At the start of inflation, the typical size of the Universe (that is, the size of a causally connected region) was H_I^{-1}, where H_I was the Hubble expansion rate at that time. During inflation, this small causally connected region was stretched by a humongous factor

$$H_I^{-1} \to e^{H_I \Delta t} H_I^{-1}, \tag{1}$$

where Δt is the duration of inflation and H_I is constant. It is useful to define $N \equiv H_I \Delta t$, which represents the number of *e-folds* that occurred during inflation. The question we would like to answer now is how many e-folds of inflationary expansion are necessary to solve either the horizon or flatness problem.

(a) While the exact energy scale at which inflation occurred is unknown, a decent guess is that it happened at the Grand Unification scale at which the electroweak and strong force unify into a single interaction. It is believed that this transition occurs when the temperature of the Universe was $T_\mathrm{GUT} \simeq 10^{15}$ GeV. If this is the case, then the Hubble rate during inflation can be estimated from the Friedmann equation as

$$H_I^2 = \frac{8\pi G}{3} \frac{\pi^2}{30} g_*(T_\mathrm{GUT}) T_\mathrm{GUT}^4. \tag{2}$$

Show that the Hubble rate during inflation in this scenario is $H_I \simeq 1.4 \times 10^{12}$ GeV. Assume only the particle content of the Standard Model. Using unit conversion, show that this corresponds to a Universe of approximate size $H_I^{-1} \simeq 1.4 \times 10^{-28}$ m at that time.

(b) Use the fact that $T_0 = 2.725$ K $= 2.348 \times 10^{-4}$ eV today to argue that the Universe has expanded by a factor of $\sim 4 \times 10^{27}$ between the end of inflation (when $T = 10^{15}$ GeV) and today. Assume that $T \propto 1/a$ always, where a is the scale factor.

(c) To solve the horizon problem, we need the initial causally connected region of size H_I^{-1} to be stretched such that the whole of the CMB last scattering surface today is causal. If the comoving radius of the last scattering surface today is $\sim 3.1 H_0^{-1}$, how many e-folds of inflation are necessary to make the last-scattering surface causal? Don't forget the amount of expansion that occurred between the end of inflation and today (i.e. the answer from part (b)). For this problem, it is useful to use the Hubble constant in natural units, $H_0 = 2.133h \times 10^{-33}$ eV, with $h = 0.674$.

(d) To solve the flatness problem, we need to explain why $|\Omega_K| < 0.001$. Writing

$$\Omega_K = \frac{\kappa}{R^2 H_0^2}, \tag{3}$$

where $\kappa = \{-1, 0, 1\}$ and R is the radius of curvature of the Universe, the constraint can be written as

$$R > 10^3 H_0^{-1}. \tag{4}$$

The flatness problem can be solved if the small causally connected region of size H_I^{-1} get stretched such that it has size $\geq R$ today. Using the above constraint on R, how many e-folds of inflation are necessary to solve the flatness problem. Again, do not forget to use your answer from part (b).

Question 3 (3 points).

Suppose that dark matter is made of black holes of mass $M = 10^{-10} M_\odot$. Assuming that the dark matter halo surrounding our Milky Way has a mass of $10^{12} M_\odot$ and radius $R = 200$ kpc, make a rough estimate for how far away you would expect the nearest such black hole to be. How does that compare to the size of the solar system?