# PHYS 480/581
# Cosmology

## Homework Assignment 6

**Due date: Friday December 9 2022, by 3:30pm at my office**

**Question 1** (6 points).

When astronomers realized that dark matter was necessary to explain the dynamical structure of galaxies, the first particle candidate that the comunity considered were neutrinos. However, it was quickly realized that neutrinos are not *cold dark matter* since they are relativistic until late into the history of the Universe.

**(a)** Since the dark matter density today is $\Omega_{\rm DM} h^2 = 0.119$, and the contribution of neutrinos to the critical density is (see HW3)

$$\Omega_\nu h^2 = \frac{\sum_i m_{\nu,i}}{94\,\text{eV}}, \tag{1}$$

what sum of neutrino masses is required for neutrinos to form all the dark matter?

**(b)** Since they are relativistic at early times, neutrinos have a lot of pressure and cannot form gravitationally bound structures. One way to see this is to compute the free-streaming length $r_{\rm fs}$ of neutrinos

$$r_{\rm fs} = \int v dt = \int_0^1 v \frac{da}{aH(a)}, \tag{2}$$

where $v = p/E$ is the neutrino speed, where $p$ is the momentum and $E$ is the energy $E = \sqrt{p^2 + m^2}$. Taking $p \simeq 3T_{\nu,0}/a$, and remembering that

$$T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,0}, \tag{3}$$

where $T_{\gamma,0} = 2.348 \times 10^{-4}$eV is the photon temperature today, compute the free-streaming length of neutrinos, taking them to have 1/3 of the the mass from part (a). You can use $H_0 = 67.5$ km/s/Mpc, $\Omega_{\rm m} = 0.3$, $\Omega_{\rm rad} = 9.1 \times 10^{-5}$, and $\Omega_\Lambda = 1 - \Omega_{\rm m} - \Omega_{\rm rad}$. How does this compare to the typical size of a Milky Way-like galaxy ($\sim 200$ kpc)? Can a galaxy like the Milky Way form out of such neutrino-made dark matter?

**Question 2** (6 points).

In class, we assumed that the gravitational potential $\Phi$ vanishes during radiation domination. In reality, the potential starts from a non-zero value at early times but quickly decays during radiation domination, reflecting the fact that radiation cannot form gravitationally bound structure. Here, we would like to explore this process in more detail. The key equation governing the evolution of the gravitational potential during radiation domination is

$$\Phi'' + \frac{4}{\eta}\Phi' + \frac{k^2}{3}\Phi = 0, \tag{4}$$

where $k$ is a constant wavenumber, prime denotes derivative with respect to $\eta$, and $\eta$ is

$$\eta = \int_0^t \frac{dt'}{a(t')}, \tag{5}$$

which is the size of the comoving causal horizon at time $t$. Here, we will use $\eta$ as our time variable. Note that the above is the equation for a damped harmonic oscillator.

**(a)** Show that if $\Phi(\eta = 0) = \Phi_{\rm p}$, then the solution to the above equation is given by

$$\Phi(\eta) = 3\Phi_{\rm p}\left(\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right). \tag{6}$$

**(b)** Plot the above solution for $\Phi$ as a function of $k\eta$, for $k\eta \in [10^{-3}, 10^3]$. Choose $\Phi_{\rm p} = 1/3$. Clearly label your axes. Confirm that $\Phi \to 0$ for $k\eta \gg 1$. This shows that the gravitational potential goes to zero after a little while during radiation domination, hence justifying our assumption of setting $\Phi = 0$ during that era.

## **Question 3** (5 points).

As we have seen, the growth of matter fluctuations $\delta_{\rm m}$ is governed by the following equation

$$\ddot{\delta}_{\rm m} + 2H\dot{\delta}_{\rm m} - \nabla^2\Phi = 0, \tag{7}$$

where $H = \dot{a}/a$ is the Hubble rate and $\Phi$ is the gravitational potential. Here, we would like to understand the evolution of matter fluctuations during dark energy domination (i.e. the late Universe). During that era, the Hubble rate $H = H_\Lambda$ is constant. Since dark energy has a large (negative) pressure, $P_\Lambda = -\rho_\Lambda$, it cannot form gravitationally bound structures, and the gravitational potential rapidly decays during dark energy domination. It is thus a good approximation to set $\Phi \sim 0$ in the above equation. Using this approximation, find the two independent solutions to the above differential equation. Can matter fluctuation grow during dark energy domination?