PHYS 480/581 - Solutions of Homework 6

Q1

a)

If neutrinos are all dark matter, then

$$\Omega_\nu h^2 = \frac{\sum_i m_{\nu,i}}{94eV} = \Omega_{DM} h^2 = 0.119 \tag{1}$$

$$\Rightarrow \sum_i m_{\nu,i} = 11.186 \;\; eV. \tag{2}$$

b)

Recall that $H(a) = H_0\sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda}$, we easily get

$$r_{fs} = \int_0^1 v \frac{da}{aH(a)} \tag{3}$$

$$= \frac{1}{H_0}\int_0^1 \frac{3T_{\nu,0}}{\sqrt{(9T_{\nu,0}^2 + m^2a^2)(\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2)}} da \tag{4}$$

$$\approx 1.936 \;\; Mpc. \tag{5}$$

The neutrino's free streaming length is roughly ten times greater than the typical size of Milky Way-like galaxy, so a galaxy like the Milky Way cannot be formed out of neutrino-made dark matter.

Q2

a)

Using the chain rule

$$\frac{d\Phi}{d\eta} = \frac{d\Phi}{d(k\eta/\sqrt{3})}\frac{d(k\eta/\sqrt{3})}{d\eta} = \frac{k}{\sqrt{3}}\frac{d\Phi}{dx},$$

$$\Rightarrow \frac{d^2\Phi}{d\eta^2} = \frac{k^2}{3}\frac{d^2\Phi}{dx^2},$$

where $x \equiv k\eta/\sqrt{3}$. The given equation becomes

$$\frac{d^2\Phi}{dx^2} + \frac{4}{x}\frac{d\Phi}{dx} + \Phi = 0. \tag{6}$$

We can check that the given solution is indeed the solution of this equation by direct substitution. We have

$$\Phi = 3\Phi_p \left(\frac{\sin x - x\cos x}{x^3} \right), \tag{7}$$

$$\frac{d\Phi}{dx} = 3\Phi_p \frac{3x\cos x + (x^2-3)\sin x}{x^4}, \tag{8}$$

and

$$\frac{d^2\Phi}{dx^2} = 3\Phi_p \frac{(x^3-12x)\cos x + (12-5x^2)\sin x}{x^5}. \tag{9}$$

From these three equations, it is straightforward to see that Eq. 6 is satisfied.

For $x << 1$ (which means $\eta \to 0$), we can do a Taylor expansion and keep the first few terms to get

$$\frac{\sin x - x\cos x}{x^3} \approx \frac{1}{x^3}\left(x - \frac{x^3}{6}\right) - \frac{1}{x^2}\left(1 - \frac{x^2}{2}\right) = \frac{1}{3}. \tag{10}$$

This implies that $\Phi(\eta = 0) = \Phi_p$ as wanted.

b)

We plot the solution of part (a) in Fig. 1. We see that $\Phi \to 0$ when $k\eta \gg 1$.

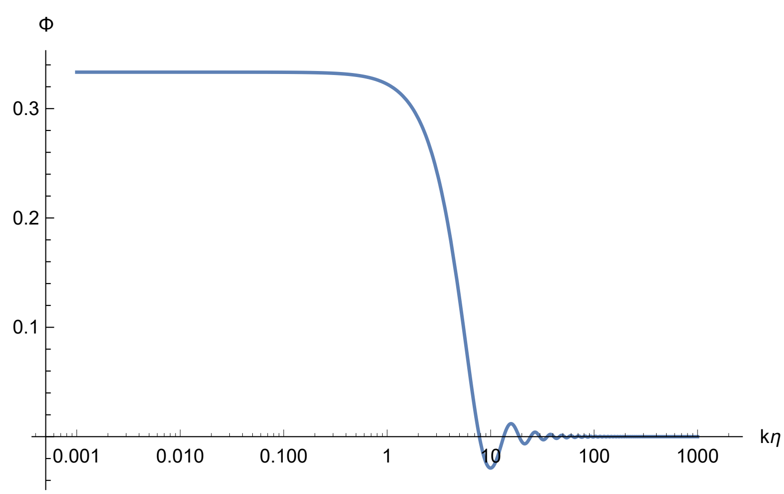

Figure 1: Gravitational potential Φ as a function of $k\eta$ during radiation domination.

Q3

If $\Phi \sim 0$, then the given equation becomes

$$\ddot{\delta}_m + 2H_\Lambda \dot{\delta}_m = 0, \tag{11}$$

whose solution is

$$\delta_m = \begin{cases} constant \\ \delta_m \propto e^{-2H_\Lambda t} \end{cases} . \tag{12}$$

These solutions can be easily obtained from the characteristic equation since our differential equation is a linear, homogeneous equation with constant coefficients (or even better, by simply guessing the solutions). We see that, in both cases, the matter fluctuation cannot grow during dark energy domination. This is intuitively expected because the rapid expansion of the Universe during the dark energy dominated era will rip things away from each other.

Alternative solution of Q2 part a

Using the chain rule

$$\frac{d\Phi}{d\eta} = \frac{d\Phi}{d(k\eta)}\frac{d(k\eta)}{d\eta} = k\frac{d\Phi}{d(k\eta)},$$

the given equation becomes

$$\Phi'' + \frac{4}{x}\Phi' + \frac{\Phi}{3} = 0, \tag{13}$$

where primes now denote derivative with respect to $x \equiv k\eta$. The general solution to this equation is (for example, see Boas's book, page 593)

$$\Phi(x) = x^{-3/2}\left[AJ_{3/2}\left(\frac{x}{\sqrt{3}}\right) + BN_{3/2}\left(\frac{x}{\sqrt{3}}\right)\right], \tag{14}$$

where $J_{3/2}$ and $N_{3/2}$ are Bessel and Neumann functions, respectively. It's easier to work with spherical Bessel and Neumann functions :

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{(2n+1)/2}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right),$$

$$n_n(x) = \sqrt{\frac{\pi}{2x}} N_{(2n+1)/2}(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right).$$

Applying these formulae to our case, we get

$$\Phi = \frac{C_1 j_1(y) + C_2 n_1(y)}{y}, \tag{15}$$

where $y \equiv x/\sqrt{3}$ and we absorbed the constants into C_1 and C_2. The spherical neumann function is

$$n_1(y) = -\frac{\sin y}{y} - \frac{\cos y}{y^2} \overset{(y<<1)}{\approx} -\frac{1}{y^2} + \mathcal{O}(y^0), \tag{16}$$

which diverges when $y \to 0$ (early times), so we shall eliminate this solution and set $C_2 = 0$. Meanwhile, the spherical Bessel function is

$$j_1(y) = \frac{\sin y}{y^2} - \frac{\cos y}{y} \overset{(y<<1)}{\approx} \frac{y}{3} + \mathcal{O}(y^3). \tag{17}$$

Applying the initial condition $\Phi(\eta = 0) = \Phi_p$, we can infer that $C_1 = 3\Phi_p$. Thus, the final solution is

$$\Phi = 3\Phi_p \left(\frac{\sin(y) - y\cos(y)}{y^3}\right) \tag{18}$$

$$= 3\Phi_p \left(\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right). \tag{19}$$

Note that $y = x/\sqrt{3} = k\eta/\sqrt{3}$.