# PHYS 480/581
# General Relativity

## Extra Problems #11

**Question 1.**

For a rotating black hole with angular momentum $J$ and mass $M$, the metric is of the Kerr form

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}\left(dtd\phi + d\phi dt\right) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(\rho^2 + a^2\right)^2 - a^2\Delta\sin^2\theta\right] d\phi^2, \tag{1}$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \tag{2}$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta \tag{3}$$

with $a = J/M$.

**(a)** Taking $M = 0$, show that these coordinates are not standard spherical coordinates, but rather *ellipsoidal coordinates*, which in Cartesian take the form

$$x = (r^2 + a^2)^{1/2}\sin\theta\cos\phi \tag{4}$$

$$y = (r^2 + a^2)^{1/2}\sin\theta\sin\phi \tag{5}$$

$$z = r\cos\theta. \tag{6}$$

Draw contours of constant $r$ and $\theta$ in the $y - z$ plane, and show that $r = 0$ corresponds to a two-dimensional disk.

**(b)** Event horizons for this metric can be solved by setting $g^{rr} = 0$. Show that this metric has two different horizons. For which values of $a$ and $M$ are these horizons physical?

**(c)** The spacetime singularity at the center here occurs at $\rho = 0$ not $r = 0$. Show that the central singularity is a ring rather than a point in this metric.