PHYS 480/581
General Relativity

Extra Problems #11 – Solutions

Question 1.

For a rotating black hole with angular momentum J and mass M, the metric is of the Kerr form

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}\,(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right] d\phi^2, \tag{1}$$

where

$$\Delta(r) = r^2 - 2GMr + a^2, \qquad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta, \tag{2}$$

with $a = J/M$.

Note: the coefficient of $d\phi^2$ in the original handout reads $\left(\rho^2+a^2\right)^2$; it should be $\left(r^2+a^2\right)^2$, as written above. (With ρ^2 the $M=0$ limit is not flat space, as part (a) shows.)

(a) Taking $M = 0$, show that these coordinates are not standard spherical coordinates, but rather *ellipsoidal coordinates*, which in Cartesian take the form

$$x = (r^2+a^2)^{1/2}\sin\theta\cos\phi, \quad y = (r^2+a^2)^{1/2}\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{3}$$

Draw contours of constant r and θ in the $y-z$ plane, and show that $r=0$ corresponds to a two-dimensional disk.

Solution. **The $M=0$ metric.** Setting $M=0$ gives $\Delta = r^2+a^2$, the $dt\,d\phi$ cross term vanishes, and

$$\left(r^2+a^2\right)^2 - a^2(r^2+a^2)\sin^2\theta = (r^2+a^2)\left(r^2+a^2\cos^2\theta\right) = (r^2+a^2)\rho^2. \tag{4}$$

The line element therefore becomes

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2+a^2}\,dr^2 + \rho^2\,d\theta^2 + (r^2+a^2)\sin^2\theta\,d\phi^2. \tag{5}$$

This is *not* the flat metric in spherical coordinates ($dr^2 + r^2 d\Omega^2$) unless $a=0$. Note, however, that the Riemann tensor of the full Kerr metric is proportional to M, so for $M=0$ this must be Minkowski space in unusual spatial coordinates. Let us show that (3) are those coordinates.

Flat metric in ellipsoidal coordinates. Define $R \equiv (r^2+a^2)^{1/2}$, so that $dR = (r/R)\,dr$. Differentiating (3),

$$\begin{aligned} dx &= \frac{r}{R}\sin\theta\cos\phi\,dr + R\cos\theta\cos\phi\,d\theta - R\sin\theta\sin\phi\,d\phi, \\ dy &= \frac{r}{R}\sin\theta\sin\phi\,dr + R\cos\theta\sin\phi\,d\theta + R\sin\theta\cos\phi\,d\phi, \\ dz &= \cos\theta\,dr - r\sin\theta\,d\theta. \end{aligned}$$

Squaring and adding the first two (the ϕ-dependence drops out through $\cos^2\phi + \sin^2\phi = 1$, and the $d\phi$ cross terms cancel):

$$dx^2 + dy^2 = \frac{r^2}{R^2}\sin^2\theta\, dr^2 + R^2\cos^2\theta\, d\theta^2 + 2r\sin\theta\cos\theta\, dr\, d\theta + R^2\sin^2\theta\, d\phi^2, \tag{6}$$

$$dz^2 = \cos^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 - 2r\sin\theta\cos\theta\, dr\, d\theta. \tag{7}$$

The $dr\, d\theta$ terms cancel, and collecting terms,

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= \frac{r^2\sin^2\theta + (r^2+a^2)\cos^2\theta}{r^2+a^2}\, dr^2 \\ &\quad + \left[(r^2+a^2)\cos^2\theta + r^2\sin^2\theta\right] d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2 \\ &= \frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\, dr^2 + \left(r^2 + a^2\cos^2\theta\right) d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2 \\ &= \frac{\rho^2}{r^2+a^2}\, dr^2 + \rho^2\, d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2. \end{aligned} \tag{8}$$

Hence $-dt^2 + dx^2 + dy^2 + dz^2$ is exactly Eq. (5): the $M = 0$ Kerr metric is Minkowski space written in the ellipsoidal (oblate spheroidal) coordinates (3). For $a = 0$ these reduce to ordinary spherical coordinates.

Contours in the y–z plane. The y–z plane is $x = 0$, i.e. $\phi = \pi/2$ (for $y > 0$) and $\phi = 3\pi/2$ (for $y < 0$). There $y = \pm R\sin\theta$ and $z = r\cos\theta$.

- *Constant r:* eliminating θ with $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1. \tag{9}$$

  These are **ellipses** with semi-major axis $\sqrt{r^2+a^2}$ along y and semi-minor axis r along z. (Rotated about the z-axis, they are oblate spheroids.) For $r \gg a$ they become circles of radius r, as for spherical coordinates.

- *Constant θ:* eliminating r via $R^2 - r^2 = a^2$,

$$\frac{y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1. \tag{10}$$

  These are **hyperbolae** (hyperboloids of one sheet in 3D). Since $r \geq 0$, each value of θ gives only one branch half: $z > 0$ for $\theta < \pi/2$ and $z < 0$ for $\theta > \pi/2$. Asymptotically ($r \gg a$) they approach the straight lines $z = y\cot\theta$, i.e. the usual polar-angle rays.

Both families share the foci $(y, z) = (\pm a, 0)$ (for the ellipses $c^2 = (r^2+a^2) - r^2 = a^2$; for the hyperbolae $c^2 = a^2\sin^2\theta + a^2\cos^2\theta = a^2$), so they form a *confocal*, mutually orthogonal grid, as shown in Fig. 1.

The surface $r = 0$. Setting $r = 0$ in (3),

$$x = a\sin\theta\cos\phi, \qquad y = a\sin\theta\sin\phi, \qquad z = 0 \quad\Longrightarrow\quad x^2 + y^2 = a^2\sin^2\theta \leq a^2, \quad z = 0. \tag{11}$$

As θ ranges over $[0, \pi]$ and ϕ over $[0, 2\pi)$, this fills the entire **disk** $x^2 + y^2 \leq a^2$ in the plane $z = 0$, not a point as in spherical coordinates. In Fig. 1 the r-ellipses flatten as $r \to 0$ and

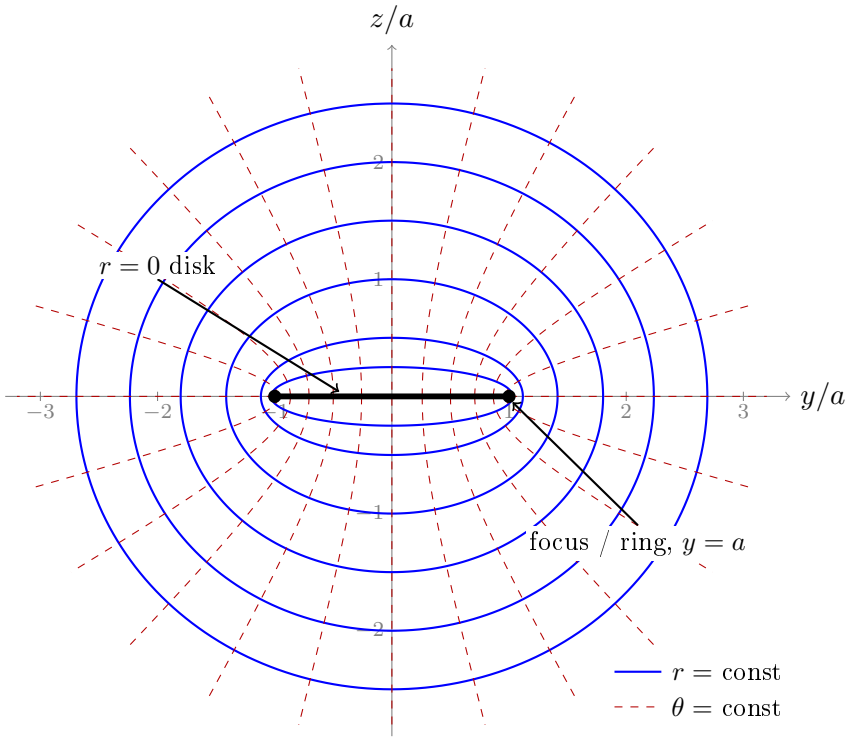


Figure 1: Ellipsoidal coordinates in the y–z plane ($a = 1$). Solid blue: $r = 0.25, 0.5, 1, 1.5, 2, 2.5$. Dashed red: $\theta = 0°, 15°, \ldots, 165°, 180°$ (the z-axis is $\theta = 0, \pi$; the equatorial line $|y| > a$ is $\theta = \pi/2$). The thick black segment $|y| \le a$, $z = 0$ is the degenerate "ellipse" $r = 0$: a cross-section of a disk of radius a.

collapse onto the segment $|y| \le a$. The center of the disk is $\theta = 0$ or π, and its rim is $\theta = \pi/2$. Each interior point is labeled twice, by θ and by $\pi - \theta$: $\theta < \pi/2$ approaches the disk from above ($z \to 0^+$) and $\theta > \pi/2$ from below ($z \to 0^-$). Equivalently, $r = 0$ is a two-sided disk of radius a.

(b) Event horizons for this metric can be solved by setting $g^{rr} = 0$. Show that this metric has two different horizons. For which values of a and M are these horizons physical?

Solution. The (r, θ) block of the metric is diagonal and decoupled from (t, ϕ), so $g^{rr} = 1/g_{rr}$:

$$g^{rr} = \frac{\Delta}{\rho^2}. \tag{12}$$

(Why $g^{rr} = 0$: the normal one-form to a surface $r = \text{const}$ is $n_\mu = \partial_\mu r$, whose norm is $g^{\mu\nu} n_\mu n_\nu = g^{rr}$. When $g^{rr} = 0$ the normal is null, so the surface is a null hypersurface. A null hypersurface can be crossed in only one direction by future-directed causal curves, which is what makes it a horizon.)

Away from the ring $\rho = 0$, the condition is

$$\Delta(r) = r^2 - 2GMr + a^2 = 0 \quad \Longrightarrow \quad \boxed{r_\pm = GM \pm \sqrt{G^2M^2 - a^2}} \tag{13}$$

Since Δ is quadratic in r, there are in general two roots:

- r_+ is the **outer horizon**, the true event horizon. It is the boundary of the region from which signals can escape to infinity.
- r_- is the **inner (Cauchy) horizon**. It is hidden behind r_+.

Both lie at positive r, because $r_+ r_- = a^2 \geq 0$ and $r_+ + r_- = 2GM > 0$. For $r_- < r < r_+$ we have $\Delta < 0$, so $g_{rr} < 0$: r becomes a timelike coordinate there and everything must move to smaller r. This is the analogue of the Schwarzschild interior.

Physical condition. The horizons are real (physical) only if the discriminant is non-negative:

$$G^2M^2 \geq a^2 \quad \Longleftrightarrow \quad a \leq GM \quad \Longleftrightarrow \quad J \leq GM^2 \ \text{(or } J \leq GM^2/c \text{ in ordinary units).} \tag{14}$$

- $a < GM$: two distinct horizons, $0 < r_- < r_+ < 2GM$. This is a black hole.
- $a = GM$ (*extremal Kerr*): the two horizons coincide at $r_+ = r_- = GM$.
- $a > GM$: $\Delta > 0$ for all r, so there is **no horizon**. The ring singularity of part (c) is then visible from infinity, a *naked singularity*. The cosmic censorship conjecture asserts that such configurations do not form from realistic gravitational collapse. So this case is usually regarded as unphysical, and astrophysical black holes satisfy $a \leq GM$.

Check: for $a \to 0$, $r_+ \to 2GM$ (the Schwarzschild horizon) and $r_- \to 0$ (merging with the central singularity).

(c) The spacetime singularity at the center here occurs at $\rho = 0$ not $r = 0$. Show that the central singularity is a ring rather than a point in this metric.

Solution. Several metric components (e.g. g_{tt} and $g_{t\phi}$) blow up as $\rho \to 0$. To confirm that this is a genuine curvature singularity rather than a coordinate artifact, we can look at the Kretschmann scalar of the Kerr metric:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48\,G^2M^2}{\rho^{12}}\left(r^2 - a^2\cos^2\theta\right)\left(\rho^4 - 16\,r^2a^2\cos^2\theta\right), \tag{15}$$

which reduces to the Schwarzschild result $48G^2M^2/r^6$ for $a = 0$ and diverges as $\rho \to 0$. Because it is a scalar, its divergence cannot be removed by a coordinate change. Hence $\rho = 0$ is a true curvature singularity.

Where is $\rho = 0$? Since $\rho^2 = r^2 + a^2\cos^2\theta$ is a sum of two non-negative terms, for $a \neq 0$

$$\rho^2 = 0 \quad \Longleftrightarrow \quad r = 0 \ \text{ and } \ \theta = \frac{\pi}{2}. \tag{16}$$

For $a = 0$, $\rho = r$ and the singularity is the familiar point $r = 0$. For $a \neq 0$ we use the geometry from part (a). (These Boyer–Lindquist coordinates are related to the Kerr–Schild "Cartesian" coordinates of the Kerr solution in the same way, $x^2 + y^2 = (r^2 + a^2)\sin^2\theta$, $z = r\cos\theta$, so the picture also holds for $M \neq 0$.) Setting $r = 0$ restricts us to the disk $x^2 + y^2 = a^2\sin^2\theta \leq a^2$, $z = 0$. Setting also $\theta = \pi/2$ picks out its boundary:

$$x = a\cos\phi, \qquad y = a\sin\phi, \qquad z = 0 \quad \Longrightarrow \quad \boxed{x^2 + y^2 = a^2, \quad z = 0} \tag{17}$$

with $\phi \in [0, 2\pi)$ free. The singularity is therefore a **ring of radius** $a = J/M$ in the equatorial plane, namely the rim of the $r = 0$ disk (the two endpoints $y = \pm a$ of the thick segment in Fig. 1, which coincide with the foci of the coordinate grid). It is not a point.

Remark: the interior of the disk ($r = 0, \theta \neq \pi/2$) is perfectly regular. There $\rho^2 = a^2 \cos^2\theta \neq 0$ and the Kretschmann scalar is finite, $-48G^2M^2/(a^6\cos^6\theta)$. An observer can pass through the disk without hitting the singularity, emerging into a region with $r < 0$ (analytically continuing r through zero).