PHYS 480/581 — General Relativity

Extra Problems #1 — Solutions

Question 1. *Equivalence-Principle deflection of starlight by the Sun.*

Conventions. SI units throughout, except where the metric is written with $c = 1$ (flagged explicitly). Newtonian potential $\Phi(r) = -GM_\odot/r < 0$.

Result.

$$\boxed{\delta_{\rm EP} \;=\; \left.\frac{2GM_\odot}{c^2 b}\right|_{b=R_\odot} \;=\; \frac{2\,(1477.1\ \text{m})}{7\times 10^{8}\ \text{m}} \;=\; 4.220\times 10^{-6}\ \text{rad} \;=\; 0.870''} \tag{1}$$

which is exactly half the general-relativistic (and observed) value $\delta_{\rm GR} = 4GM_\odot/c^2R_\odot \simeq 1.74''$. The aspect of the Equivalence Principle (EP) that is not respected is its *strict locality*: the calculation glues an infinite chain of local inertial frames together using a *globally Euclidean* spatial geometry.

1 Setup and the physical input from the EP

Place the Sun at the origin. The photon travels in the $+\hat{x}$ direction along the line $y = b$, with impact parameter $b \simeq R_\odot$ (grazing incidence). Write $r = \sqrt{x^2 + b^2}$.

The single piece of physics taken from the EP is the *universality of free fall applied to light.* Consider an Einstein elevator momentarily at rest in the solar field at the photon's location, together with its freely falling counterpart. In the falling frame the photon moves in a straight line — that is the EP: locally, gravity is abolished. The static frame accelerates upward relative to the falling one with acceleration $+\boldsymbol{g}$, where $\boldsymbol{g} = -\nabla\Phi$. Therefore, *as measured in the static frame*, the photon acquires transverse velocity at the rate

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} \;=\; -\frac{\partial\Phi}{\partial y}, \tag{2}$$

independently of the photon's speed and of its energy. That universality is precisely the EP content, and it is what licenses treating a massless particle with the same $-\nabla\Phi$ that a Newtonian test mass would feel.

Assumptions :

1. *Weak field*: $GM_\odot/(c^2R_\odot) \simeq 2.1\times 10^{-6} \ll 1$.
2. *Unperturbed (Born) trajectory*: evaluate the force along the straight line $y = b$ with $v_x \simeq c$, so $\mathrm{d}t = \mathrm{d}x/c$. Corrections are $\mathcal{O}(\delta^2)$.
3. *Small angle*: $\sin\delta \simeq \delta$.
4. *Static, spherically symmetric, isolated deflector*; the solar quadrupole and the Sun's motion are neglected.

2 The calculation

On the unperturbed path,

$$\frac{\partial\Phi}{\partial y}\bigg|_{y=b} = \frac{GM_\odot\, y}{r^3}\bigg|_{y=b} = \frac{GM_\odot\, b}{(x^2+b^2)^{3/2}}, \tag{3}$$

so that, using $\mathrm{d}t = \mathrm{d}x/c$ from Eq. (2),

$$v_y = \int_{-\infty}^{\infty} \frac{\mathrm{d}v_y}{\mathrm{d}t}\,\mathrm{d}t = -\frac{GM_\odot\, b}{c}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+b^2)^{3/2}}. \tag{4}$$

The elementary integral is

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+b^2)^{3/2}} = \left[\frac{x}{b^2\sqrt{x^2+b^2}}\right]_{-\infty}^{\infty} = \frac{2}{b^2}, \tag{5}$$

and therefore

$$v_y = -\frac{2GM_\odot}{c\,b}, \qquad \Longrightarrow \qquad \sin\delta = \frac{|v_y|}{c} = \frac{2GM_\odot}{c^2 b}. \tag{6}$$

Dimensional check. $GM_\odot/c^2$ is a length (given as 1477.1 m), so $GM_\odot/(c^2 b)$ is dimensionless, as required for $\sin\delta$. Likewise v_y in Eq. (6) carries units $[\mathrm{m}\cdot\mathrm{m\,s^{-1}}/\mathrm{m}] = \mathrm{m\,s^{-1}}$.

Numerics. With $b = R_\odot = 7\times10^5$ km $= 7\times10^8$ m,

$$\delta_{\mathrm{EP}} = \frac{2\times 1477.1}{7\times10^8} = 4.2203\times10^{-6}\ \mathrm{rad}\ \times\ \frac{180}{\pi}\times 3600\ \frac{''}{\mathrm{rad}} = 0.8705'', \tag{7}$$

which is the required result. Using modern values, $GM_\odot/c^2 = 1476.6$ m and $R_\odot = 6.957\times10^8$ m, one finds $0.876''$, and $1.751''$ for the GR prediction — the standard quoted numbers.

Consistency of the small-angle step. Since $\delta \sim 4\times10^{-6}$, we have $\sin\delta = \delta$ to a part in 10^{11}, and the Born approximation likewise fails only at $\mathcal{O}(\delta^2)$. Neither approximation is responsible for the factor of 2.

3 Which aspect of the EP is not respected

Answer: its strict locality.

The EP is a statement about an *infinitesimal* neighbourhood: in a freely falling frame small compared with the curvature scale, physics is that of special relativity and there is *no deflection at all.* Any nonzero deflection is therefore, by construction, an accumulation of the *relative orientation of successive local inertial frames* along the ray — and that relative orientation is fixed by spacetime curvature, not by the EP.

The calculation of Sec. 2 quietly does two things the EP does not license:

1. It uses *one global frame* spanning a region of size $\gg R_\odot$, i.e. vastly larger than any local inertial frame. Over that region tidal (curvature) effects are *not* negligible — indeed they are the entire effect.

2. In gluing the local results together, it assumes (t, x, y) are Minkowskian coordinates: that $\mathrm{d}x$ is proper distance and that space is *Euclidean.* Only the time–time part of the geometry is allowed to respond to Φ.

Remark (a common misdiagnosis). It is tempting to say "the calculation is wrong because it is Newtonian." That is not quite the right diagnosis. The numerical answer does coincide with the Newtonian corpuscular result, but the reason it falls short by a factor of 2 is specifically that the EP was applied *non-locally*, and the non-local extrapolation smuggled in a flat spatial metric. The EP is not wrong; it is simply not enough. It constrains the *local* structure of spacetime, whereas determining the *global* geometry requires field equations — which is exactly why Einstein needed 1915 and not merely 1911.

4 Historical note (optional)

The $0.87''$ answer is not a pedagogical straw man: it is Einstein's own 1911 prediction, obtained from the EP alone before he possessed field equations. He quoted $0.83''$ using the solar parameters then in use.

- A. Einstein, *Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes*, Ann. Phys. (Leipzig) **35** [340], 898–908 (1911). DOI: 10.1002/andp.19113401005; ADS bibcode `1911AnP...340..898E`.
- J. G. von Soldner (1801), *Berliner Astronomisches Jahrbuch für das Jahr 1804*, pp. 161–172, obtained the identical Newtonian value a century earlier. Provenance and the well-known factor-of-two misprint are documented in T. Sauer, *Soldner, Einstein, Gravitational Light Deflection and Factors of Two*, Ann. Phys. (Berlin) (2021), DOI: 10.1002/andp.202100203.
- The doubled GR value was confirmed by F. W. Dyson, A. S. Eddington & C. Davidson, Phil. Trans. R. Soc. Lond. A **220**, 291–333 (1920), reporting $1.61'' \pm 0.30''$ from the 1919 Sobral/Príncipe eclipse.