PHYS 480/581
General Relativity

Extra Problems #4

Question 1.

The metric for a two-sphere is

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where a the the radius of the sphere. This metric tells you how to compute distances on the surface of a sphere of constant radius. Now, imagine that there is a rank-4 tensor $\boldsymbol{R}$ living on this two-sphere, with the following properties

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \qquad R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}, \tag{2}$$

and

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}. \tag{3}$$

where here $\rho, \sigma, \mu, \nu = \theta$ or ϕ. Say I give you one component of this tensor

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta. \tag{4}$$

(a) Using the properties given above and Eq. (4), find all 16 components of $\boldsymbol{R}$ with all four lower indices, i.e., $R_{\rho\sigma\mu\nu}$.
Solution:
Using the antisymmetry of $\boldsymbol{R}$ in either its two first indices or its two last, we know that any component of $\boldsymbol{R}$ where either the two first indices are the same, or the two last indices are the same will automatically vanish. This means that

$$R_{\theta\theta\phi\phi} = R_{\theta\theta\theta\phi} = R_{\theta\theta\phi\theta} = R_{\theta\theta\theta\theta} = R_{\phi\phi\theta\theta} = R_{\phi\phi\theta\phi} = R_{\phi\phi\phi\theta} = R_{\phi\phi\phi\phi} = 0, \tag{5}$$

$$R_{\theta\phi\theta\theta} = R_{\phi\theta\theta\theta} = R_{\phi\theta\phi\phi} = R_{\theta\phi\phi\phi} = 0. \tag{6}$$

This leaves four non-vanishing components

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\lambda}R^{\lambda}{}_{\phi\theta\phi} \\ &= g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} \\ &= a^2\sin^2\theta, \end{aligned} \tag{7}$$

and

$$R_{\phi\theta\theta\phi} = -a^2\sin^2\theta, \quad R_{\theta\phi\phi\theta} = -a^2\sin^2\theta, \quad R_{\phi\theta\phi\theta} = a^2\sin^2\theta. \tag{8}$$

(b) Define the second-rank tensor $\tilde{\boldsymbol{R}}$ with components given by

$$\tilde{R}_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{9}$$

Find the four components of $\tilde{\boldsymbol{R}}$.
Solution:

First notice that $\tilde{R}_{\mu\nu}$ is symmetric in his two indices since

$$\begin{aligned}\tilde{R}_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu}\\ &= g^{\alpha\beta}R_{\beta\mu\alpha\nu}\\ &= g^{\alpha\beta}R_{\alpha\nu\beta\mu}\\ &= g^{\beta\alpha}R_{\alpha\nu\beta\mu}\\ &= R^{\beta}{}_{\nu\beta\mu}\\ &= \tilde{R}_{\nu\mu}. \end{aligned} \tag{10}$$

Then,

$$\tilde{R}_{\phi\theta} = \tilde{R}_{\theta\phi} = g^{\alpha\beta}R_{\beta\theta\alpha\phi} = g^{\theta\theta}R_{\theta\theta\theta\phi} + g^{\phi\phi}R_{\phi\theta\phi\phi} = 0. \tag{11}$$

We are then left with

$$\tilde{R}_{\phi\phi} = g^{\alpha\beta}R_{\beta\phi\alpha\phi} = g^{\theta\theta}R_{\theta\phi\theta\phi} + g^{\phi\phi}R_{\phi\phi\phi\phi} = \frac{1}{a^2}a^2\sin^2\theta + 0 = \sin^2\theta, \tag{12}$$

and

$$\tilde{R}_{\theta\theta} = g^{\alpha\beta}R_{\beta\theta\alpha\theta} = g^{\theta\theta}R_{\theta\theta\theta\theta} + g^{\phi\phi}R_{\phi\theta\phi\theta} = 0 + \frac{1}{a^2\sin^2\theta}a^2\sin^2\theta = 1. \tag{13}$$

(c) Find the trace of $\tilde{\boldsymbol{R}}$, $\tilde{R}^{\mu}{}_{\mu}$.
Solution:
We have

$$\tilde{R}^{\mu}{}_{\mu} = g^{\mu\alpha}\tilde{R}_{\alpha\mu} = g^{\theta\theta}\tilde{R}_{\theta\theta} + g^{\phi\phi}\tilde{R}_{\phi\phi} = \frac{1}{a^2} + \frac{1}{a^2\sin^2\theta}\sin^2\theta = \frac{2}{a^2}. \tag{14}$$