

PHYS 480/581 General Relativity

Extra Problems #5

Question 1.

Consider the following Lagrangian for an electrically charged complex scalar field $\phi(x)$ (with charge e) in Minkowski space

$$\mathcal{L} = \eta^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (1)$$

where $D_\mu \equiv \partial_\mu + ieA_\mu$ is the gauge covariant derivative, A_μ is the gauge field, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength (curvature) tensor.

(a) Show that the above Lagrangian is invariant under the gauge transformation

$$\phi(x) \rightarrow e^{ie\theta(x)} \phi(x), \quad A_\mu \rightarrow A_\mu - \partial_\mu \theta(x). \quad (2)$$

(b) In extra Problem #3, you showed that

$$\frac{\partial(F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = 4F^{\mu\nu}. \quad (3)$$

Use this result together with the Euler-Lagrange equation for A_ν

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0, \quad (4)$$

to derive 2 of Maxwell's equations in this theory [they obviously will have the form $\partial_\mu F^{\mu\nu} = -J^\nu$, so the question is really about what does the 4-current J^ν look like in this theory.]

(c) By Noether's theorem, any theory that has a symmetry (here a $U(1)$ symmetry) has an associated conserved quantity. Show that this conserved quantity is the current J^μ you found in part (b), that is, show that,

$$\partial_\mu J^\mu = 0. \quad (5)$$