

PHYS 480/581 — General Relativity

Extra Problems #5 — Solutions

Gauge invariance and the current of a charged scalar field

Setup and conventions

The theory is

$$\mathcal{L} = \eta^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad D_\mu \equiv \partial_\mu + ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1)$$

Throughout, $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, and $\hbar = c = 1$.¹ The charge e is a real constant, the gauge field A_μ is a real vector field, and the gauge parameter $\theta(x)$ is a real function. The scalar ϕ is complex, so ϕ and ϕ^* count as two independent fields when we vary the action; equivalently, one may vary the real and imaginary parts of ϕ separately.

It will be convenient to note once and for all that complex conjugation commutes with ∂_μ and flips the sign of the ieA_μ term, since A_μ and e are real:

$$(D_\mu \phi)^* = \partial_\mu \phi^* - ieA_\mu \phi^*. \quad (2)$$

Indices on D are raised with η in the usual way, $D^\mu \equiv \eta^{\mu\nu} D_\nu$, so the first term of (1) can be written compactly as $(D_\mu \phi)^* (D^\mu \phi)$.

One honest bookkeeping remark before starting: in the $(-, +, +, +)$ signature the scalar term of (1) unpacks as $-|D_0 \phi|^2 + |D_i \phi|^2$, i.e. it carries the opposite overall sign to the one that makes the scalar's kinetic energy positive. That is purely a matter of how the problem normalizes $\mathcal{L}$; flipping it would flip the overall sign of the current J^ν in part (b) and change nothing else. We therefore work with $\mathcal{L}$ exactly as given in (1).

(a) Gauge invariance

The gauge transformation is

$$\phi(x) \rightarrow \phi'(x) = e^{ie\theta(x)} \phi(x), \quad A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu \theta(x). \quad (3)$$

Note first *why* the covariant derivative is needed. The naive derivative of ϕ does not simply pick up a phase,

$$\partial_\mu \phi' = \partial_\mu (e^{ie\theta} \phi) = e^{ie\theta} (\partial_\mu \phi + ie \partial_\mu \theta \phi), \quad (4)$$

because θ depends on position. The unwanted term $ie \partial_\mu \theta \phi$ would spoil the invariance of $\partial^\mu \phi^* \partial_\mu \phi$. The role of A_μ is precisely to cancel it.

¹In fact none of the manipulations in parts (a)–(c) depend on the signature. The metric enters only through the raising of indices, $D^\mu \equiv \eta^{\mu\nu} D_\nu$, and every step below contracts one upper index against one lower index. The signature is needed only when components are unpacked — for instance $\partial^0 = -\partial_t$ in this convention.

Key lemma: $D_\mu \phi$ transforms covariantly. Applying the transformed covariant derivative to the transformed field,

$$\begin{aligned} D'_\mu \phi' &= (\partial_\mu + ieA'_\mu) \phi' = \partial_\mu (e^{ie\theta} \phi) + ie(A_\mu - \partial_\mu \theta) e^{ie\theta} \phi \\ &= e^{ie\theta} \left(\partial_\mu \phi + \underbrace{ie \partial_\mu \theta \phi}_{\text{from } \partial_\mu e^{ie\theta}} + ieA_\mu \phi - \underbrace{ie \partial_\mu \theta \phi}_{\text{from } -ie \partial_\mu \theta} \right) \\ &= e^{ie\theta} (\partial_\mu \phi + ieA_\mu \phi), \end{aligned} \quad (5)$$

that is,

$$\boxed{D_\mu \phi \rightarrow e^{ie\theta(x)} D_\mu \phi} \quad (6)$$

The two $\partial_\mu \theta$ terms cancel, so $D_\mu \phi$ transforms exactly like ϕ itself: it picks up the same overall phase and nothing else. This is the entire content of the word “covariant” in “gauge covariant derivative.”

The scalar term is invariant. Conjugating (6) gives $(D_\mu \phi)^* \rightarrow e^{-ie\theta} (D_\mu \phi)^*$, so the phases cancel pairwise:

$$\eta^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) \rightarrow \eta^{\mu\nu} \left(e^{-ie\theta} (D_\mu \phi)^* \right) \left(e^{ie\theta} (D_\nu \phi) \right) = \eta^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi). \quad (7)$$

Note that $e^{ie\theta(x)}$ is just a number at each point, so it commutes freely with everything here; no expansion in small θ is required, and the result is exact for arbitrary $\theta(x)$.

The Maxwell term is invariant. The field strength itself is unchanged:

$$F_{\mu\nu} \rightarrow \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - (\partial_\mu \partial_\nu \theta - \partial_\nu \partial_\mu \theta) = F_{\mu\nu}, \quad (8)$$

since partial derivatives commute on any smooth function, $\partial_\mu \partial_\nu \theta = \partial_\nu \partial_\mu \theta$. Hence $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is trivially invariant.

Adding the two pieces,

$$\mathcal{L} [\phi', A'_\mu] = \mathcal{L} [\phi, A_\mu], \quad (9)$$

so the Lagrangian — and therefore the action and the field equations — are invariant under the local $U(1)$ gauge transformation (3). ■

(b) Maxwell's equations and the current

To vary with respect to A_ν it is easiest to expand the scalar term explicitly, using (2):

$$\begin{aligned} \eta^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) &= (\partial^\mu \phi^* - ieA^\mu \phi^*) (\partial_\mu \phi + ieA_\mu \phi) \\ &= \partial^\mu \phi^* \partial_\mu \phi + ieA^\mu (\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi) + e^2 A^\mu A_\mu |\phi|^2. \end{aligned} \quad (10)$$

The middle term is the expected “current dotted into the gauge field” interaction; the last term, quadratic in A_μ , is required by gauge invariance and is often called the seagull term.

The two required derivatives. Only the last two terms of (10) contain A_ν undifferentiated. Differentiating with respect to A_ν (and remembering that $A^\mu A_\mu = \eta^{\mu\lambda} A_\mu A_\lambda$ is quadratic, which produces the factor of 2):

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = ie (\phi \partial^\nu \phi^* - \phi^* \partial^\nu \phi) + 2e^2 A^\nu |\phi|^2. \quad (11)$$

Only the Maxwell term contains $\partial_\mu A_\nu$, so with the result quoted from Extra Problem #3,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4} \frac{\partial (F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4} (4F^{\mu\nu}) = -F^{\mu\nu}. \quad (12)$$

Euler–Lagrange equation. Substituting (11) and (12) into $\partial \mathcal{L} / \partial A_\nu - \partial_\mu [\partial \mathcal{L} / \partial(\partial_\mu A_\nu)] = 0$ gives

$$\frac{\partial \mathcal{L}}{\partial A_\nu} + \partial_\mu F^{\mu\nu} = 0 \quad \implies \quad \partial_\mu F^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial A_\nu} \equiv -J^\nu, \quad (13)$$

so that, comparing with the form $\partial_\mu F^{\mu\nu} = -J^\nu$ stated in the problem, the four-current is

$$\boxed{J^\nu = \frac{\partial \mathcal{L}}{\partial A_\nu} = ie (\phi \partial^\nu \phi^* - \phi^* \partial^\nu \phi) + 2e^2 A^\nu |\phi|^2} \quad (14)$$

These are two of Maxwell’s equations: the $\nu = 0$ component is Gauss’s law and the $\nu = i$ components are the Ampère–Maxwell law, with J^ν as the source. (The other two, $\nabla \cdot \mathbf{B} = 0$ and Faraday’s law, follow identically from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and are not field equations here.)

Covariant form. Result (14) looks like it treats ∂^ν and A^ν separately, but the two pieces combine into a single manifestly gauge-covariant expression. Using (2),

$$\begin{aligned} ie [\phi (D^\nu \phi)^* - \phi^* (D^\nu \phi)] &= ie [\phi (\partial^\nu \phi^* - ie A^\nu \phi^*) - \phi^* (\partial^\nu \phi + ie A^\nu \phi)] \\ &= ie (\phi \partial^\nu \phi^* - \phi^* \partial^\nu \phi) + e^2 A^\nu |\phi|^2 + e^2 A^\nu |\phi|^2, \end{aligned} \quad (15)$$

which is exactly (14). Hence

$$J^\nu = ie [\phi (D^\nu \phi)^* - \phi^* (D^\nu \phi)] = 2e \operatorname{Im}(\phi^* D^\nu \phi). \quad (16)$$

Two checks on (16):

- J^ν is **real**, as a physical current must be: $\phi^* D^\nu \phi - \phi (D^\nu \phi)^* = \phi^* D^\nu \phi - (\phi^* D^\nu \phi)^*$ is $2i$ times the imaginary part of $\phi^* D^\nu \phi$, and multiplying by $-ie$ leaves something real.
- J^ν is **gauge invariant**, as the source of the gauge-invariant $F^{\mu\nu}$ must be: under (3), $\phi^* \rightarrow e^{-ie\theta} \phi^*$ and $D^\nu \phi \rightarrow e^{ie\theta} D^\nu \phi$ by the lemma (6), so the phases cancel.

Structurally, J^ν is the Klein–Gordon (“probability”) current of the complex scalar, multiplied by the charge e and with ∂^ν promoted to D^ν . Its overall sign is fixed by the problem’s convention $\partial_\mu F^{\mu\nu} = -J^\nu$, which in the $(-, +, +, +)$ signature is the standard form of Maxwell’s equations: in this signature $F^{0i} = E^i$ and $J^\mu = (\rho, \mathbf{J})$, so the $\nu = 0$ component of $\partial_\mu F^{\mu\nu} = -J^\nu$ is $-\nabla \cdot \mathbf{E} = -\rho$, i.e. Gauss’s law.

(c) Current conservation

By Noether's theorem, the symmetry of part (a) must be accompanied by a conserved current. The relevant case is a *constant* θ : then $\partial_\mu \theta = 0$, so A_μ is untouched and the symmetry is the global $U(1)$ phase rotation $\phi \rightarrow e^{ie\theta} \phi$ acting on the scalar alone. What follows verifies directly that the current J^μ found in part (b) is the conserved quantity that Noether's theorem promises, using nothing but the equation of motion of the scalar field.

Field equation for the scalar. Vary (1) with respect to ϕ^* . From (2),

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = \eta^{\mu\nu} (-ieA_\mu) (D_\nu \phi) = -ieA_\mu D^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} = D^\mu \phi, \quad (17)$$

so the Euler–Lagrange equation $\partial \mathcal{L} / \partial \phi^* - \partial_\mu [\partial \mathcal{L} / \partial (\partial_\mu \phi^*)] = 0$ reads

$$-ieA_\mu D^\mu \phi - \partial_\mu (D^\mu \phi) = 0 \quad \implies \quad (\partial_\mu + ieA_\mu) D^\mu \phi = D_\mu D^\mu \phi = 0. \quad (18)$$

This is the gauge-covariant (massless) Klein–Gordon equation. Its complex conjugate, $(D_\mu D^\mu \phi)^* = 0$, is the equation one gets by varying ϕ .

Divergence of the current. The useful form of (18) is

$$\partial_\mu (D^\mu \phi) = -ieA_\mu D^\mu \phi, \quad \text{and conjugating,} \quad \partial_\mu (D^\mu \phi)^* = +ieA_\mu (D^\mu \phi)^*. \quad (19)$$

Now differentiate (16) with the product rule:

$$\partial_\mu J^\mu = ie [\partial_\mu \phi (D^\mu \phi)^* + \phi \partial_\mu (D^\mu \phi)^* - \partial_\mu \phi^* (D^\mu \phi) - \phi^* \partial_\mu (D^\mu \phi)]. \quad (20)$$

Using (19) on the second and fourth terms,

$$\partial_\mu J^\mu = ie \left[\underbrace{\partial_\mu \phi (D^\mu \phi)^* + ieA_\mu \phi (D^\mu \phi)^*}_{\text{group into } D_\mu \phi} - \underbrace{\partial_\mu \phi^* (D^\mu \phi) + ieA_\mu \phi^* (D^\mu \phi)}_{\text{group into } -(D_\mu \phi)^*} \right]. \quad (21)$$

Indeed $\partial_\mu \phi + ieA_\mu \phi = D_\mu \phi$ and $-(\partial_\mu \phi^* - ieA_\mu \phi^*) = -(D_\mu \phi)^*$, so the four terms collapse to

$$\partial_\mu J^\mu = ie [(D_\mu \phi) (D^\mu \phi)^* - (D_\mu \phi)^* (D^\mu \phi)] = 0, \quad (22)$$

because the two products are identical — each factor is an ordinary (commuting) complex function, and both terms contract the same pair of indices. Hence

$$\boxed{\partial_\mu J^\mu = 0} \quad (23)$$

as required. ■

Remark: a one-line consistency check. Conservation is also forced by the Maxwell equation itself. Taking ∂_ν of $\partial_\mu F^{\mu\nu} = -J^\nu$,

$$\partial_\nu J^\nu = -\partial_\nu \partial_\mu F^{\mu\nu} = 0, \quad (24)$$

since $\partial_\nu \partial_\mu$ is symmetric in $(\mu\nu)$ while $F^{\mu\nu}$ is antisymmetric, and the contraction of a symmetric with an antisymmetric tensor vanishes. This is quicker, but it is a *consistency requirement*: it shows the gauge field cannot couple to a non-conserved current at all. The direct calculation above is the substantive statement — that the matter sector, using its own equation of motion (18), actually supplies a conserved current.

Remark: physical content. Writing $J^\mu = (\rho, \mathbf{J})$, the statement $\partial_\mu J^\mu = 0$ is the continuity equation $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$. Integrating over all space and discarding the surface term (fields falling off at infinity) gives a time-independent total charge

$$Q = \int d^3x J^0, \quad \frac{dQ}{dt} = 0, \quad (25)$$

which is the conserved Noether charge of the $U(1)$ symmetry: the total electric charge of the scalar field.