PHYS 480/581
General Relativity

Extra Problems #8

Question 1.

Consider the action of a massive free scalar field ϕ

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}m^2\phi^2\right], \tag{1}$$

where m is the mass of the field and g is the determinant of the metric.

(a) Using the identity $\ln(\det M) = \mathrm{Tr}(\ln M)$ (where M is a square non-singular matrix), show that

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}. \tag{2}$$

(b) Use the definition of the inverse metric $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$ to show that

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. \tag{3}$$

(c) Use the above to show that

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{4}$$

(d) Using the definition of the stress-energy tensor,

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}}, \tag{5}$$

compute $T_{\mu\nu}$ for the scalar field ϕ.

(e) Show that this stress energy tensor satisfies the the weak ($T_{\mu\nu}t^\mu t^\nu \geq 0$ for timelike vector t^μ) and the null energy ($T_{\mu\nu}l^\mu l^\nu \geq 0$ for null vectors l^μ) conditions.