PHYS 480/581
General Relativity

Extra Problems #9

Question 1.

Let's take a very long thin solid rod of mass density λ (per unit length) extending along the z-axis (we can neglect the pressure of the rod). The rod is moving in the z direction with a small coordinate speed $V \ll 1$, making it essentially a static source. Using the non-relativistic and weak-field limit (i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$) of the geodesic equation

$$\frac{d^2x^i}{dt^2} \approx \frac{1}{2}\delta^{il}\partial_l h_{tt} + \delta^{il}\left(\partial_l h_{tj} - \partial_j h_{lt}\right)v^j, \tag{1}$$

we would like to compute the equation of motion for a non-relativistic particle propagating near this rod. Here, $\vec{x}$ is the 3D position of the particle, and $\vec{v}$ is its velocity. In the weak, static limit, the Einstein equation admits the solution

$$h_{\beta\nu}(\vec{r}) = 2\int d^3r_{\rm s}\frac{G(2T_{\beta\nu} - \eta_{\beta\nu}T)}{|\vec{r} - \vec{r}_{\rm s}|}, \tag{2}$$

where the integral runs over where the stress-energy tensor has support (i.e. where the matter/energy is).

(a) Assuming that the stress-energy tensor admits a perfect fluid form $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ (with $p = 0$), compute the relevant source terms

$$2T_{tt} - \eta_{tt}T \qquad \text{and} \qquad 2T_{ti} - \eta_{ti}T,\ (i = 1, 2, 3), \tag{3}$$

in terms of λ and V. Here, $T = g^{\mu\nu}T_{\mu\nu}$.

(b) Write down the formal solutions for h_{tt} and h_{tj} (easiest to use cylindrical coordinates (r, ϕ, z) for integration) and compute their derivatives $\partial_r h_{tt}$ and $\partial_r h_{tj}$.

(c) Write down the (x, y, z) components of Eq. (1) and show that the equation of motion for the non-relativistic particle can be written in the form

$$\frac{d^2\vec{x}}{dt^2} = \vec{E}_G + \vec{v} \times 4\vec{B}_G, \tag{4}$$

where $\vec{E}_G$ is the gravitoelectric field and $\vec{B}_G$ is the gravitomagnetic field. Write down the exact expressions for these fields in terms of λ and V.

(d) Sketch the motion of a particle initially at a distance L from the rod and at rest.