PHYS 480/581
General Relativity

Extra Problems #9 Solutions

Question 1.

Let's take a very long thin solid rod of mass density λ (per unit length) extending along the z-axis (we can neglect the pressure of the rod). The rod is moving in the z direction with a small coordinate speed $V \ll 1$, making it essentially a static source. Using the non-relativistic and weak-field limit (i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$) of the geodesic equation

$$\frac{d^2x^i}{dt^2} \approx \frac{1}{2}\delta^{il}\partial_l h_{tt} + \delta^{il}\left(\partial_l h_{tj} - \partial_j h_{lt}\right)v^j, \tag{1}$$

we would like to compute the equation of motion for a non-relativistic particle propagating near this rod. Here, $\vec{x}$ is the 3D position of the particle, and $\vec{v}$ is its velocity. In the weak, static limit, the Einstein equation admits the solution

$$h_{\beta\nu}(\vec{r}) = 2\int d^3r_{\rm s}\frac{G(2T_{\beta\nu} - \eta_{\beta\nu}T)}{|\vec{r} - \vec{r}_{\rm s}|}, \tag{2}$$

where the integral runs over where the stress-energy tensor has support (i.e. where the matter/energy is).

(a) Assuming that the stress-energy tensor admits a perfect fluid form $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ (with $p = 0$), compute the relevant source terms

$$2T_{tt} - \eta_{tt}T \qquad \text{and} \qquad 2T_{ti} - \eta_{ti}T,\ (i = 1, 2, 3), \tag{3}$$

in terms of λ and V. Here, $T = g^{\mu\nu}T_{\mu\nu}$.

(b) Write down the formal solutions for $h_{tt}(r)$ and $h_{tj}(r)$ (easiest to use cylindrical coordinates (s, ϕ, z) for integration) and compute their derivatives $\partial_r h_{tt}$ and $\partial_r h_{tj}$.

(c) Write down the (x, y, z) components of Eq. (1) and show that the equation of motion for the non-relativistic particle can be written in the form

$$\frac{d^2\vec{x}}{dt^2} = \vec{E}_G + \vec{v} \times 4\vec{B}_G, \tag{4}$$

where $\vec{E}_G$ is the gravitoelectric field and $\vec{B}_G$ is the gravitomagnetic field. Write down the exact expressions for these fields in terms of λ and V.

(d) Sketch the motion of a particle initially at a distance L from the rod and at rest.

Solution:

We would like to compute the equation of motion for a non-relativistic particle propagating near this rod. We thus need to compute $\partial_l h_{tt}$ and $\partial_l h_{tj}$. From the formal solution to the Poisson equation above, we have

$$h_{tt}(r) = 2\int d^3 r_{\rm s} \frac{G(2T_{tt} - \eta_{tt}T)}{|\vec{r} - \vec{r}_{\rm s}|} \tag{5}$$

With $p = 0$, we have $T_{\mu\nu} = \rho u_\mu u_\nu$, and therefore the trace $T = \rho g^{\mu\nu} u_\mu u_\nu = -\rho$. Since the rod is along the z axis, we can write its energy density as $\rho = \lambda\delta^{(2)}(\vec{s})$, where $\vec{s}$ is the 2D polar vector. It is easiest to do the integral in cylindrical coordinates (s, ϕ, z). In this case, $|\vec{r} - \vec{r}_{\rm s}| = \sqrt{r^2 + z^2}$, and $2T_{tt} - \eta_{tt}T = \lambda\delta^{(2)}(\vec{s})$. We thus get

$$h_{tt}(r) = 2G\lambda \int_\infty^\infty dz \frac{1}{\sqrt{r^2 + z^2}}, \tag{6}$$

where we have used the delta function $\delta^{(2)}(\vec{s})$ to perform the s and ϕ integrals. Similarly, with $2T_{tz} - \eta_{tz}T = 2T_{tz} = 2\rho u_t u_z \approx 2\rho(-1)V = -2V\lambda\delta^{(2)}(\vec{s})$,

$$h_{tz}(r) = -4G\lambda V \int_\infty^\infty dz \frac{1}{\sqrt{r^2 + z^2}} = -2V h_{tt}(r), \tag{7}$$

with $h_{tx} = h_{ty} = 0$, since both $T_{tx} = 0$ and $T_{ty} = 0$. Now the integrals we have above are technically infinite, but this is ok as what we want is the spatial derivative of these integrals. In particular,

$$\begin{aligned}
\partial_r h_{tt} &= 2G\lambda \int_\infty^\infty dz \partial_r \left(\frac{1}{\sqrt{r^2 + z^2}} \right) \\
&= -2G\lambda \int_\infty^\infty dz \frac{r}{(r^2 + z^2)^{3/2}} \\
&= -2G\lambda r \left(\frac{z}{r^2\sqrt{r^2 + z^2}} \right)_{-\infty}^{\infty} \\
&= -2G\lambda r \left(\frac{1}{r^2} - \frac{-1}{r^2} \right) \\
&= -\frac{4G\lambda}{r}.
\end{aligned} \tag{8}$$

Similarly, we get

$$\partial_r h_{tz}(r) = \frac{8G\lambda V}{r} \tag{9}$$

Remembering that $\partial_x r = x/r$, $\partial_y r = y/r$, and $\partial_z r = 0$ in cylindrical, we thus have

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{1}{2}\partial_x h_{tt} + \left(\partial_x h_{tj} - \partial_j h_{xt}\right) v^j \\
&= \frac{1}{2}\partial_r h_{tt}\partial_x r + v^z \partial_r h_{tz}\partial_x r \\
&= -\frac{2G\lambda x}{r^2} + \frac{8G\lambda V v^z x}{r^2}.
\end{aligned} \tag{10}$$

Similarly

$$\frac{d^2y}{dt^2} = -\frac{2G\lambda y}{r^2} + \frac{8G\lambda V v^z y}{r^2}. \tag{11}$$

Finally

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{1}{2}\partial_z h_{tt} + \left(\partial_z h_{tj} - \partial_j h_{zt}\right) v^j \\ &= -\partial_x h_{zt} v^x - \partial_y h_{zt} v^y \\ &= -v^x \partial_r h_{zt} \partial_x r - v^y \partial_r h_{zt} \partial_y r \\ &= -\frac{8G\lambda V}{r^2}\left(x v^x + y v^y\right). \end{aligned} \tag{12}$$

We note that this is very similar to the Lorentz force law for electromagnetism. In fact, we can write the above as

$$\frac{d^2\vec{x}}{dt^2} = \vec{E}_G + \vec{v} \times 4\vec{B}_G, \tag{13}$$

with

$$\vec{E}_G = -\frac{2G\lambda}{r}\hat{r}, \qquad \vec{B}_G = -\frac{2G\lambda}{r}(\vec{V} \times \hat{r}). \tag{14}$$

Up to a sign and the factor of 4 for the "magnetic" term, compare this from the electric and magnetic field of a moving line charge,

$$\vec{E} = \frac{2\lambda}{4\pi\epsilon_0 r}\hat{r}, \qquad \vec{B} = \frac{2\lambda}{4\pi\mu_0 r}(\vec{V} \times \hat{r}). \tag{15}$$

This is why the contribution from the second term in Eq. (1) above is often referred to as the gravitomagnetic term.