

PHYS 480/581 General Relativity

Worksheet #4
Monday September 14 2026

Question 1.

Consider the following completely antisymmetric object called the Levi-Civita symbol

$$\tilde{\epsilon}_{ij} = \begin{cases} +1 & \text{if } ij = 12, \\ -1 & \text{if } ij = 21, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

By definition, the Levi-Civita symbol has the components specified above in *any* (right-handed) coordinate systems. The main question we want to answer here is whether $\tilde{\epsilon}_{ij}$ is a tensor. If it is, it should transform under a coordinate change $x \rightarrow x'(x)$ according to

$$\tilde{\epsilon}'_{kl} \stackrel{?}{=} \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} \tilde{\epsilon}_{ij} = J^i_k J^j_l \tilde{\epsilon}_{ij}, \quad (2)$$

where we have introduced the notation J^i_k for the Jacobian, with

$$J^i_k \equiv \frac{\partial x^i}{\partial x'^k}. \quad (3)$$

(a) To get some intuition, let's write the Jacobian matrix as

$$J^i_k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad (4)$$

Use this form to evaluate the right-hand side of Eq. (2) and fill in the following table

k	l	$\tilde{\epsilon}_{ij} J^i_k J^j_l$
1	1	
1	2	
2	1	
2	2	

(b) Use the result from part (a) to write down a general form for the contraction

$$J^i_k J^j_l \tilde{\epsilon}_{ij}. \quad (5)$$

Given this, is $\tilde{\epsilon}_{ij}$ a tensor?

Question 2.

The metric for the Poincaré half-plane

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad y > 0. \quad (6)$$

Now, imagine that there is a rank-4 tensor $\mathbf{R}$ living in this half-plane, with the following properties

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \quad R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}, \quad (7)$$

and

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}. \quad (8)$$

where here $\rho, \sigma, \mu, \nu = x$ or y . Say I give you one component of this tensor

$$R^x_{yxy} = -1/y^2. \quad (9)$$

(a) Using the properties given above and Eq. (9), find all 16 components of $\mathbf{R}$ with all four lower indices, i.e., $R_{\rho\sigma\mu\nu}$.

(b) Compute the contraction

$$R \equiv g^{\mu\nu} R_{\mu\alpha\nu}^{\alpha} \quad (10)$$

and show that it is a constant.