PHYS 480/581
General Relativity

Worksheet #8

Wednesday November 18 2026

Question 1.

Orbiting point masses: Consider two point masses of equal mass M on a circular orbit of radius $R(t)$ with angular frequency Ω in the xy-plane. Note that Ω is related to M and R via the standard Keplerian relations (not needed here). Let's labeled the two masses a and b. Their instantaneous position is given by

$$\vec{r}_a = (R(t)\cos\Omega t, R(t)\sin\Omega t, 0), \tag{1}$$

$$\vec{r}_b = (-R(t)\cos\Omega t, -R(t)\sin\Omega t, 0). \tag{2}$$

Since these are point masses, their mass density is described by Dirac delta function

$$T^{tt}(t,\mathbf{x}) = \rho(t,\mathbf{x}) = M\delta(z)\left(\delta(x - R(t)\cos\Omega t)\delta(y - R(t)\sin\Omega t) + \delta(x + R(t)\cos\Omega t)\delta(y + R(t)\sin\Omega t)\right). \tag{3}$$

To compute the gravitational wave signal, we first need to compute the reduced quadrupole moment

$$\mathcal{I}^{ij}(t) = \int (x'^i x'^j - \frac{1}{3}\eta^{ij} r'^2)\rho(t,\mathbf{x}')d^3x'. \tag{4}$$

To get you started, here is the computation of the xx componet (dropping the primes to reduce clutter)

$$\begin{aligned}
\mathcal{I}^{xx}(t) &= \int (x^2 - \frac{1}{3}(x^2+y^2+z^2))\rho(t,\mathbf{x})dxdydz \\
&= M\int (x^2 - \frac{1}{3}(x^2+y^2))\left(\delta(x - R(t)\cos\Omega t)\delta(y - R(t)\sin\Omega t) + \delta(x + R(t)\cos\Omega t)\delta(y + R(t)\sin\Omega t)\right)dxdy \\
&= 2M\left(R^2(t)\cos^2\Omega t - \frac{1}{3}(R^2(t)\cos^2\Omega t + R^2(t)\sin^2\Omega t)\right) \\
&= 2MR^2(t)\left(\cos^2\Omega t - \frac{1}{3}\right) \\
&= 2MR^2(t)\left(\frac{\cos 2\Omega t}{2} + \frac{1}{6}\right) \\
&= \frac{MR^2(t)}{3}\left(3\cos 2\Omega t + 1\right).
\end{aligned} \tag{5}$$

(a) Following the example above, compute the other components of the reduced quadrupole tensor and show that is is equal to

$$\mathcal{I}^{ij}(t) = \frac{MR^2(t)}{3}\begin{bmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{6}$$

(b) Using the solution to the Einstein equation we derived in class

$$H^{ij}(t,\mathbf{r}) = \frac{2G}{r}\ddot{\mathcal{I}}^{ij}(t-r). \tag{7}$$

show that the gravitational wave signal at a distance r from the source is given by

$$H^{ij}(t,\mathbf{r}) = \frac{8GMR^2(t_r)\Omega^2}{r}\begin{bmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8}$$

where $t_r \equiv t - r$. In deriving the above, assume that the $R(t)$ is changing very slowly (i.e. much slower than the rotation period $T = 2\pi/\Omega$). We will check this approximation in the homework. Note that the frequency of oscillation of the gravitational wave is always twice the orbital frequency of the two point masses.

(c) For which direction of propagation is this wave transverse?