

PHYS 480/581 General Relativity

Homework Assignment 5

Due date: Monday 09/21/2026 8pm, submitted electronically on UNM Canvas

Question 1 (6 points).

Consider adding to the Lagrangian of electromagnetism an additional term of the form $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = \tilde{F}_{\rho\sigma} F^{\rho\sigma}$. Here, $\tilde{F}_{\rho\sigma}$ is called the Hodge dual of the standard electromagnetic field strength $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$, and $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol.

- (a) Express $\mathcal{L}'$ in terms of the electric $\vec{E}$ and magnetic $\vec{B}$ fields.
- (b) Using the Euler-Lagrange equation for the vector potential A^μ

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}_{\text{em}}}{\partial (\partial_\mu A_\nu)} \right) = 0, \quad (1)$$

and the solution from the extra problem #3, show that including $\mathcal{L}'$ does not affect Maxwell's equations. Here, $\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu + \mathcal{L}'$. Can you provide a deep reason why?

Question 2 (5 points).

A quantity that we will need in the near future is the determinant of the metric $\det(g)$. In a 4-dimensional spacetime, this determinant can be computed via the relation

$$\tilde{\epsilon}^{\mu\nu\alpha\beta} g_{\mu\gamma} g_{\nu\delta} g_{\alpha\sigma} g_{\beta\rho} = \det(g) \tilde{\epsilon}_{\gamma\delta\sigma\rho}, \quad (2)$$

which is just the standard expression for the determinant of a 4×4 matrix.

- (a) Show that for a diagonal metric, the above expression simply gives that $\det(g)$ is just the product of the diagonal elements of $g_{\mu\nu}$.
- (b) While $\det(g)$ is a scalar function, it is not a *Lorentz* scalar, meaning that it takes different values in different inertial reference frames. Using the transformation for the Levi-Civita symbol under a coordinate transform $x^\mu \rightarrow x'^\mu$

$$\tilde{\epsilon}'_{\alpha\beta\gamma\delta} = \left| \frac{\partial x'}{\partial x} \right| \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\gamma} \frac{\partial x^\rho}{\partial x'^\delta} \tilde{\epsilon}_{\mu\nu\sigma\rho}, \quad (3)$$

show that

$$\det(g') = \left| \frac{\partial x'}{\partial x} \right|^{-2} \det(g), \quad (4)$$

where $\left| \frac{\partial x'}{\partial x} \right|$ is the determinant of the Jacobian matrix for the coordinate transformation $x^\mu \rightarrow x'^\mu$.

(c) Use Eqs. (3) and (4) above to show that

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{\det(g)} \tilde{\epsilon}_{\alpha\beta\gamma\delta} \quad (5)$$

is an actual tensor, unlike $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$.