# Physics 480/581
# General Relativity

## Homework Assignment 7 Solutions

**Question 1** (5 points).

We saw in class that the Riemann tensor is defined as the commutator of covariant derivatives

$$[\nabla_\mu, \nabla_\nu]V^\alpha = R^\alpha{}_{\beta\mu\nu}V^\beta, \tag{1}$$

where $\mathbf{V}$ is an arbitrary 4-vector. Note that in flat spacetime, the covariant derivatives reduce to standard partial derivatives (which commute), implying that $R^\alpha{}_{\beta\mu\nu} = 0$ in that case, as it should.

**(a)** Since $\nabla_\mu V^\alpha$ is a standard (1,1) tensor, show that

$$\begin{aligned}\nabla_\mu(\nabla_\nu V^\alpha) = \partial_\mu\partial_\nu V^\alpha + (\partial_\mu\Gamma^\alpha_{\beta\nu})V^\beta + \Gamma^\alpha_{\beta\nu}\partial_\mu V^\beta - \Gamma^\sigma_{\nu\mu}\partial_\sigma V^\alpha \\ - \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\gamma\sigma}V^\gamma + \Gamma^\alpha_{\delta\mu}\partial_\nu V^\delta + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\lambda\nu}V^\lambda.\end{aligned} \tag{2}$$

**(b)** Use the result from part **(a)** to derive the standard relationship between the Riemann curvature tensor and the Christoffel connection

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\beta\mu}. \tag{3}$$

**Solutions:**

**(a)** Since $\nabla_\nu a^\alpha$ is a perfectly fine $(1,1)$ tensor, we have

$$\begin{aligned}\nabla_\mu(\nabla_\nu a^\alpha) &= \partial_\mu(\nabla_\nu a^\alpha) + \Gamma^\alpha_{\mu\beta}(\nabla_\nu a^\beta) - \Gamma^\sigma_{\mu\nu}(\nabla_\sigma a^\alpha) \\ &= \partial_\mu(\partial_\nu a^\alpha + \Gamma^\alpha_{\nu\beta}a^\beta) + \Gamma^\alpha_{\mu\beta}(\partial_\nu a^\beta + \Gamma^\beta_{\nu\lambda}a^\lambda) - \Gamma^\sigma_{\mu\nu}(\partial_\sigma a^\alpha + \Gamma^\alpha_{\sigma\gamma}a^\gamma) \\ &= \partial_\mu\partial_\nu a^\alpha + (\partial_\mu\Gamma^\alpha_{\nu\beta})a^\beta + \Gamma^\alpha_{\nu\beta}\partial_\mu a^\beta + \Gamma^\alpha_{\mu\beta}\partial_\nu a^\beta + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\lambda}a^\lambda - \Gamma^\sigma_{\mu\nu}\partial_\sigma a^\alpha - \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\gamma}a^\gamma \\ &= \partial_\mu\partial_\nu a^\alpha + (\partial_\mu\Gamma^\alpha_{\beta\nu})a^\beta + \Gamma^\alpha_{\beta\nu}\partial_\mu a^\beta + \Gamma^\alpha_{\delta\mu}\partial_\nu a^\delta - \Gamma^\sigma_{\nu\mu}\partial_\sigma a^\alpha + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\nu\lambda}a^\lambda - \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\gamma\sigma}a^\gamma,\end{aligned} \tag{4}$$

where we have used $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$, and relabeled some dummy indices.

**(b)** Using the above with $\mu \leftrightarrow \nu$, we have that

$$\nabla_\nu(\nabla_\mu a^\alpha) = \partial_\nu\partial_\mu a^\alpha + (\partial_\nu\Gamma^\alpha_{\beta\mu})a^\beta + \Gamma^\alpha_{\beta\mu}\partial_\nu a^\beta + \Gamma^\alpha_{\delta\nu}\partial_\mu a^\delta - \Gamma^\sigma_{\mu\nu}\partial_\sigma a^\alpha + \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\mu\lambda}a^\lambda - \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\gamma\sigma}a^\gamma.$$

And therefore

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]a^\alpha &= (\partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu})a^\beta + \Gamma^\alpha_{\beta\nu}\partial_\mu a^\beta + \Gamma^\alpha_{\delta\mu}\partial_\nu a^\delta - \Gamma^\alpha_{\beta\mu}\partial_\nu a^\beta - \Gamma^\alpha_{\delta\nu}\partial_\mu a^\delta \\
&\quad + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\nu\lambda}a^\lambda - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\mu\lambda}a^\lambda \\
&= (\partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu})a^\beta + \Gamma^\alpha_{\beta\nu}\partial_\mu a^\beta + \Gamma^\alpha_{\beta\mu}\partial_\nu a^\beta - \Gamma^\alpha_{\beta\mu}\partial_\nu a^\beta - \Gamma^\alpha_{\beta\nu}\partial_\mu a^\beta \\
&\quad + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\nu\lambda}a^\lambda - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\mu\lambda}a^\lambda \\
&= (\partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu})a^\beta + (\Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\nu\beta} - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\mu\beta})a^\beta \\
&= (\partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\nu\beta} - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\mu\beta})a^\beta \\
&= (\partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\delta}\Gamma^\delta_{\beta\nu} - \Gamma^\alpha_{\nu\delta}\Gamma^\delta_{\beta\mu})a^\beta \\
&= R^\alpha{}_{\beta\mu\nu}a^\beta, \qquad (5)
\end{aligned}$$

and thus

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\delta}\Gamma^\delta_{\beta\nu} - \Gamma^\alpha_{\nu\delta}\Gamma^\delta_{\beta\mu}. \qquad (6)$$

**Question 2** (5 points).

Consider the following metric for a two-dimensional space

$$ds^2 = dp^2 + e^{2p/p_0}dq^2, \qquad (7)$$

where $p_0$ is a constant with units of length.

**(a)** Argue that the only nonzero Christoffel symbols for this metric are $\Gamma^p_{qq}$ and $\Gamma^q_{pq} = \Gamma^q_{qp}$, and evaluate these symbols.

**(b)** Is this space curved? Prove your assertion.

**Solutions:**
We have the two dimensional metric

$$ds^2 = dp^2 + e^{2p/p_0}dq^2 \qquad (8)$$

**(a)** From the metric above, we observe that only $g_{qq}$ is nonconstant, and the only first first derivative of the metric is

$$\partial_p g_{qq} = \frac{2}{p_0}e^{2p/p_0} \qquad (9)$$

Since the metric is diagonal, this term will appear only in the Christoffel symbols involving 1 $p$ index and 2 $q$ indices. There are two indepesndent Christoffels that have this structure

$$\begin{aligned}
\Gamma^q_{pq} = \Gamma^q_{qp} &= \frac{1}{2}g^{qq}\left(\partial_q g_{pq} + \partial_p g_{qq} - \partial_q g_{qp}\right) \\
&= \frac{1}{2}e^{-2p/p_0}\frac{2}{p_0}e^{2p/p_0} \\
&= \frac{1}{p_0}, \qquad (10)
\end{aligned}$$

$$\begin{aligned}\Gamma^p_{qq} &= \frac{1}{2}g^{pp}\left(\partial_q g_{qp} + \partial_q g_{pq} - \partial_p g_{qq}\right)\\ &= -\frac{1}{2}\frac{2}{p_0}e^{2p/p_0}\\ &= -\frac{e^{2p/p_0}}{p_0}\end{aligned} \tag{11}$$

**(b)** Does this metric represent a curved or flat space? Let's compute the Riemann curvature tensor. The only nonzero Riemann tensor element is given by

$$\begin{aligned}R_{pqpq} &= g_{p\alpha}R^{\alpha}{}_{qpq}\\ &= g_{pp}R^{p}{}_{qpq} + g_{pq}R^{q}{}_{qpq}\\ &= g_{pp}\left(\partial_p\Gamma^p_{qq} - \partial_q\Gamma^p_{qp} + \Gamma^p_{p\alpha}\Gamma^\alpha_{qq} - \Gamma^p_{q\sigma}\Gamma^\sigma_{qp}\right)\\ &= \partial_p\Gamma^p_{qq} - \Gamma^p_{qq}\Gamma^q_{qp}\\ &= \partial_p\left(-\frac{e^{2p/p_0}}{p_0}\right) + \frac{e^{2p/p_0}}{p_0}\frac{1}{p_0}\\ &= e^{2p/p_0}\left(-\frac{2}{p_0^2} + \frac{1}{p_0^2}\right)\\ &= -\frac{e^{2p/p_0}}{p_0^2} \neq 0.\end{aligned} \tag{12}$$

So this space is curved. While we are at it (not necessary for full marks), let's compute the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = g^{\alpha\beta}R_{\beta\mu\alpha\nu}. \tag{13}$$

Thus

$$\begin{aligned}R_{pp} &= g^{\alpha\beta}R_{\beta p\alpha p}\\ &= g^{qq}R_{qpqp}\\ &= -e^{-2p/p_0}\frac{e^{2p/p_0}}{p_0^2}\\ &= -\frac{1}{p_0^2}.\end{aligned} \tag{14}$$

$$\begin{aligned}R_{qq} &= g^{\alpha\beta}R_{\beta q\alpha q}\\ &= g^{pp}R_{pqpq}\\ &= -\frac{e^{2p/p_0}}{p_0^2}.\end{aligned} \tag{15}$$

$$\begin{aligned}R_{pq} = R_{qp} &= g^{\alpha\beta}R_{\beta p\alpha q}\\ &= 0.\end{aligned} \tag{16}$$

And finally the Ricci scalar is

$$\begin{aligned} R &= g^{\mu\nu} R_{\mu\nu} \\ &= g^{pp} R_{pp} + g^{qq} R_{qq} \\ &= -\frac{1}{p_0^2} + e^{-2p/p_0}(-\frac{e^{2p/p_0}}{p_0^2}) \\ &= -\frac{2}{p_0^2}. \end{aligned} \tag{17}$$

So the curvature is constant everywhere in this space.

**Question 3** (4 points).

Consider the metric

$$ds^2 = -dt^2 + a(t)^2 \left[dx^2 + dy^2 + dz^2\right] \tag{18}$$

where $a(t)$ in an increasing function of time (and $x, y, z$ are Euclidean spatial coordinates), which describes a homogeneous and isotropic expanding universe.

**(a)** Compute the Christoffel symbols for this metric.
**Solutions:**
It is advantageous to write the above metric as

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \tag{19}$$

where $i, j = 1, 2, 3$, and $\delta_{ij}$ is the 3D spatial Kronecker delta. The only derivative of the metric which is nonzero is

$$\partial_0 g_{ij} = 2a\dot{a}\delta_{ij}, \tag{20}$$

where $\dot{a} = da/dt$, and the zero index represents the time component. Since the metric is diagonal, the only nonzero Christoffel connections will be those involving one 0 index and two spatial indices. There are only a few possibilities:

$$\begin{aligned} \Gamma^0_{ij} &= \frac{1}{2} g^{00} (\partial_i g_{j0} + \partial_j g_{0i} - \partial_0 g_{ij}) \\ &= -\frac{1}{2}(-2a\dot{a}\delta_{ij}) \\ &= a\dot{a}\delta_{ij}. \end{aligned} \tag{21}$$

$$\begin{aligned} \Gamma^i_{0j} &= \frac{1}{2} g^{ik} (\partial_0 g_{jk} + \partial_j g_{k0} - \partial_k g_{0j}) \\ &= \frac{1}{2a^2} \delta^{ik} 2a\dot{a}\delta_{jk} \\ &= \frac{\dot{a}}{a}\delta^i_j = \Gamma^i_{j0}. \end{aligned} \tag{22}$$

All other Christoffels are zero.

**(b)** Write down the equation of motion of a point particle in this metric in the absence of external forces and derive the time dependence of its physical three-velocity. Comment on your result.
**Solutions:**
The geodesic takes the form

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{23}$$

Since the four-velocity is defined as $u^\mu = dx^\mu/d\tau$, this can be rewritten as

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0. \tag{24}$$

The $i$ component is

$$\frac{du^i}{d\tau} + \Gamma^i_{0j}u^0u^j + \Gamma^i_{j0}u^ju^0 = \frac{du^i}{d\tau} + 2\frac{\dot{a}}{a}u^0u^i = 0. \tag{25}$$

The 0 component is

$$\frac{du^0}{d\tau} + \Gamma^0_{ij}u^iu^j = \frac{du^0}{d\tau} + a\dot{a}\delta_{ij}u^iu^j = 0. \tag{26}$$

Now, the squared magnitude of the **spatial** components of the four-velocity is

$$u^2 = g_{ij}u^iu^j = a^2(t)\delta_{ij}u^iu^j. \tag{27}$$

Equation (26) can then be written as

$$\frac{du^0}{d\tau} + \frac{\dot{a}}{a}u^2 = 0. \tag{28}$$

Since the four-velocity obeys the normalization

$$u_\mu u^\mu = g_{\mu\nu}u^\mu u^\nu = -(u^0)^2 + g_{ij}u^iu^j = -(u^0)^2 + u^2 = -1, \tag{29}$$

we have the relationship (differentiation everything with respect to $\tau$)

$$2u^0\frac{du^0}{d\tau} = 2u\frac{du}{d\tau}. \tag{30}$$

Equation (28) can then be written as

$$\frac{u}{u^0}\frac{du}{d\tau} + \frac{\dot{a}}{a}u^2 = 0. \tag{31}$$

Now,

$$\frac{du}{d\tau} = \frac{du}{dt}\frac{dt}{d\tau} = \frac{du}{dt}u^0, \tag{32}$$

and we are left with

$$\frac{du}{dt} + \frac{\dot{a}}{a}u = 0, \qquad (u \neq 0). \tag{33}$$

Rewriting this as

$$\frac{1}{a}\frac{d}{dt}(ua) = 0, \tag{34}$$

we obtain

$$u \propto \frac{1}{a(t)}. \tag{35}$$

Since $a(t)$ is an increasing function of time, we thus obtain that $u$ decays as a function of time. If you want, we can also express your answer in terms of the coordinate (or ordinary) velocity $v^i = u^i/u^0$. We have

$$v^2 = g_{ij}v^i v^j = a^2(t)\delta_{ij}\frac{u^i u^j}{(u^0)^2} = \frac{u^2}{(u^0)^2} = \frac{u^2}{1+u^2}, \tag{36}$$

where we have used Eq. (29) above. Writing $u = a_0/a(t)$, where $a_0$ is a positive normalization constant, we thus get

$$v = \frac{a_0/a(t)}{\sqrt{1+(a_0/a(t))^2}}. \tag{37}$$

We see that $v \to 1$ as $a(t) \to 0$ and $v \to 0$ for $a(t) \gg 1$. So even a particle that is ultrarelativistic at early times will eventually become nonrelativistic at some later time in an expanding universe.