PHYS 480/581
General Relativity

Homework Assignment 8

Due date: Monday 10/19/2026 8pm, submitted electronically on UNM Canvas

Question 1 (5 points).

The Einstein-Hilbert action in n spacetime dimensions is given by

$$S_{\rm H} = \int d^n x \sqrt{-g}\, R = \int d^n x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}, \tag{1}$$

where R is the Ricci scalar and g is the determinant of the metric. By varying this action with respect to the inverse metric $g^{\mu\nu}$ and setting $\delta S_{\rm H} = 0$, one can derive Einstein's equation. This variation leads to 3 terms

$$\delta S_{\rm H} = \int d^n x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} + \int d^n x \sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu} + \int d^n x R\, \delta\sqrt{-g}. \tag{2}$$

The first term is actually a total derivative (can you show that?) and thus does not contribute to the equation of motion. The second term is already of the form we want (i.e. a variation with respect to the inverse metric). The third term is what we need to focus on.

(a) Using the definition of the inverse metric $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$, show that the variation of the metric and of the inverse metric are related as follows

$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}. \tag{3}$$

(b) Use the identity $\ln(\det M) = \mathrm{Tr}(\ln M)$ (where M is a square non-singular matrix) and the result from part (a) to show that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}. \tag{4}$$

(c) Use the above results and set $\delta S_{\rm H} = 0$ to derive Einstein's equation in vacuum

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0. \tag{5}$$

Question 2 (5 points).

The Lagrangian density for electromagnetism in curved spacetime is

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu\right), \tag{6}$$

where J^μ is the electric four-current and g is the determinant of the metric. Using the definition of the stress-energy tensor

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{7}$$

where $S = \int d^4x\mathcal{L}$ is the action, compute the stress energy tensor for electromagnetism. You may find some of the results from Question 1 useful.

Question 3 (6 points).

Moore Problem 20.10. This problem requires the concept of local orthonormal frame (LOF). Denoting the basis vector in the LOF as $\hat{\mathbf{e}}_{(a)}$, where we use latin indices to denote components of tensors in the LOF. Our standard coordinate basis vectors $\hat{\mathbf{e}}_{(\mu)}$ can be written in terms of the LOF basis vectors as

$$\hat{\mathbf{e}}_{(\mu)} = e_\mu{}^a\hat{\mathbf{e}}_{(a)} \tag{8}$$

where the component of $\hat{\mathbf{e}}_{(\mu)}$ in the direction of LOF basis vector $\hat{\mathbf{e}}_{(a)}$, denoted $e_\mu{}^a$ is referred as the "vielbein" or "tetrad". The key properties of the vielbeins is that they are orthonormal with respect to the standard coordinate basis metric

$$g_{\mu\nu}e^\mu{}_a e^\nu{}_b = \eta_{ab}, \tag{9}$$

where $e^\mu{}_a$ are the inverse vielbeins, defined via $e^\mu{}_a e_\nu{}^a = \delta^\mu_\nu$ and $e_\mu{}^a e^\mu{}_b = \delta^a_b$. Note that Moore denotes the inverse vielbeins as $e^\mu{}_a = (\boldsymbol{o}_a)^\mu$. Using Eq. (9), we can derive an alternative relation between the vielbein and its inverse

$$e_\mu{}^a = g_{\mu\nu}\eta^{ab}e^\nu{}_b \tag{10}$$

Now, if you know the components V^μ of some vector $\mathbf{V}$ in our standard coordinate basis, the components of $\mathbf{V}$ in the LOF basis are simply

$$V^a = e_\mu{}^a V^\mu. \tag{11}$$

Similarly, if you know the stress-energy tensor components in a coordinate basis, its components in the LOF basis will simply be

$$T^{ab} = e_\mu{}^a e_\nu{}^b T^{\mu\nu}, \tag{12}$$

where T^{ab} here is what Moore calls T^{ab}_{obs}. Note that substituting Eq. (10) into Eq. (12) yield Eq. (20.34) in Moore. Discussing the different energy conditions in a LOF makes the problem a lot more tractable.