

PHYS 480/581: General Relativity

Properties of the Riemann and Ricci tensors

Prof. Cyr-Racine
(Dated: September 30, 2026)

I. PROPERTIES OF THE RIEMANN TENSOR

[**A remarkable commutator**] Last time, we defined the Riemann curvature tensor via the relation

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma, \quad (1)$$

which is valid for spacetime without torsion, that is, those for which $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. Let's think for a moment how remarkable this relation is. On the right-hand side, we have double covariant derivatives applied on the components of the vector $\mathbf{V}$ which is equal on the left to a linear relation with the components of $\mathbf{V}$, *independent* of the derivatives of V^σ . From this relation, the physical meaning $R^\rho_{\sigma\mu\nu}$ emerges: **it describes how the σ component of a vector gets projected into its ρ component when parallel transported around an infinitesimal loop in the plane given by the μ, ν coordinates.**

[**Nonlinear in the metric**] The Riemann tensor can be written in terms of the Christoffel connection as

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\sigma} \Gamma^\sigma_{\beta\mu}. \quad (2)$$

Note that the products of Christoffel connections in the Riemann tensor means it is nonlinear in the metric. As a consequence, GR is a nonlinear theory, which is one of the reason why it is so difficult to find solutions to Einstein's equation.

[**Properties of the Riemann tensor**] From its definition in terms of the commutator in Eq. (1), it's clear that Riemann tensor is antisymmetric in its two last indices

$$R^\alpha_{\beta\mu\nu} = -R^\alpha_{\beta\nu\mu}. \quad (3)$$

Note that this is also true with all indices lowered

$$R_{\alpha\beta\nu\mu} = -R_{\alpha\beta\mu\nu}. \quad (4)$$

To find the other properties of the Riemann tensor, we can specialize to a locally inertial frame (LIF), where the derivative of the metric at some point p is zero and the metric there is Minkowski. This means that the Christoffel connection coefficients are zero at point p , although their derivatives (which involves second derivatives of the metric) are generally nonzero there. At this point, the Riemann tensor takes the form [**Box 19.1**]

$$R_{\alpha\beta\mu\nu} = \frac{1}{2} (\partial_\beta \partial_\mu g_{\alpha\nu} + \partial_\alpha \partial_\nu g_{\beta\mu} - \partial_\beta \partial_\nu g_{\alpha\mu} - \partial_\alpha \partial_\mu g_{\beta\nu}) \quad (5)$$

From this, we can read off the other properties of the Riemann tensor. It is antisymmetric in its two first indices

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \quad (6)$$

and it is symmetric under interchange of the first pair of indices with the second

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}. \quad (7)$$

You can also show that

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0. \quad (8)$$

The 3 equations above were derived in a LIF, but they are all valid tensor equations, and thus must be valid in all possible frames. The Riemann tensor also satisfies the Bianchi identity,

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0, \quad (9)$$

which is essentially a geometric consistency relation [box 19.4].

[Number of independent components] Given these symmetries, we can show that in an n -dimensional spacetime, the Riemann tensor has

$$\frac{1}{12}n^2(n^2 - 1) \quad (10)$$

independent components. In four spacetime dimensions, this means that there are 20 independent components. In 2 space dimensions (like the surface of a two-sphere), there is a single independent component of the Riemann tensor.

II. CONTRACTIONS OF THE RIEMANN TENSOR

We can construct the Ricci tensor by contracting the first and third indices

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}. \quad (11)$$

Using the symmetry of the Riemann tensor, it is straightforward to show that the Ricci tensor is symmetric in its two indices

$$R_{\mu\nu} = R_{\nu\mu}. \quad (12)$$

Let's show that

$$\begin{aligned} R_{\mu\nu} &= R^\alpha_{\mu\alpha\nu} \\ &= g^{\alpha\beta} R_{\beta\mu\alpha\nu} \\ &= g^{\alpha\beta} R_{\alpha\nu\beta\mu} \\ &= g^{\beta\alpha} R_{\alpha\nu\beta\mu} \\ &= R^\beta_{\nu\beta\mu} \\ &= R_{\nu\mu}. \end{aligned} \quad (13)$$

We can also take the trace of the Ricci tensor to form the Ricci scalar (or curvature scalar) R

$$R = R^\mu_{\mu} = g^{\mu\nu} R_{\mu\nu}. \quad (14)$$

R is real Lorentz scalar, meaning that it is the same in every coordinate systems.

III. INDEPENDENT COMPONENTS OF THE RIEMANN TENSOR

Given these two properties of the Riemann tensor

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \quad (15)$$

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}, \quad (16)$$

we need $\alpha \neq \beta$ and $\mu \neq \nu$ in order for the Riemann tensor to be nonzero. In n dimensions, the pair $\alpha\beta$ can admit $n(n-1)$ values. However, not all of these pairs are independent since switching the order of $\alpha\beta$ only flips the sign of the Riemann tensor. So there are $n(n-1)/2$ independent pairs of $\alpha\beta$. The same applies for the $\mu\nu$ pair. So far, we thus have

$$\frac{n^2(n-1)^2}{4} \quad (17)$$

independent components of the Riemann. Now we also have to take into account the symmetry property

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}. \quad (18)$$

This essentially halves the number of independent components, except for the $n(n-1)/2$ “diagonal” elements for which $\alpha\beta = \mu\nu$. So subtracting the diagonal elements from the total, dividing by two, and then adding back the diagonal elements, we obtain

$$\frac{1}{2} \left(\frac{n^2(n-1)^2}{4} - \frac{n(n-1)}{2} \right) + \frac{n(n-1)}{2} = \frac{1}{8} \left(n(n-1)(n(n-1)+2) \right). \quad (19)$$

The last tricky part is to take into account this final property

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0. \quad (20)$$

The above can only be non-zero if $\alpha \neq \beta \neq \mu \neq \nu$. This means that the above only provide additional constraints if $n \geq 4$. Given that all indices need to be different, and that we have complete permutation symmetry, the number of constraints is

$$\frac{n(n-1)(n-2)(n-3)}{4!}, \quad (21)$$

which automatically ensures that no new constraints arise if $n < 4$. Putting everything together we get

$$\frac{1}{8} \left(n(n-1)(n(n-1)+2) \right) - \frac{n(n-1)(n-2)(n-3)}{24} = \frac{1}{12} n^2 (n^2 - 1). \quad (22)$$