

PHYS 480/581: General Relativity

Tensors

Prof. Cyr-Racine
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I. REVIEW OF VECTORS AND DUAL VECTORS

We have seen before that a (four-)vector is an object that lives in the tangent space $T_p M$ of some spacetime point p of some manifold M , as illustrated in Fig. 1. Dual vectors (or one-forms) live in what we call the dual vector space (or cotangent space) $T_p^* M$ at point p . I remind you that dual vectors are linear maps from the original vector space $T_p M$ to the real numbers; that is, if $\omega \in T_p^* M$ is a dual vector, it acts as

$$\omega(a\mathbf{V} + b\mathbf{W}) = a\omega(\mathbf{V}) + b\omega(\mathbf{W}) \in \mathbb{R}, \quad (1)$$

where $\mathbf{V}$, $\mathbf{W}$ are vectors, and a , b are real numbers. Again, you should think of a dual vector as a “machine” that “eats” a vector and returns a real number. We have also seen that we can expand a vector in terms of a coordinate basis $\mathbf{V} = V^\mu \mathbf{e}_{(\mu)}$. We can of course also define a coordinate basis for dual vectors such that $\omega = \omega_\mu \mathbf{e}^{(\mu)}$. The dual basis vector $\mathbf{e}^{(\mu)}$ are defined from the coordinate basis vectors as

$$\mathbf{e}^{(\nu)}(\mathbf{e}_{(\mu)}) = \delta_\mu^\nu. \quad (2)$$

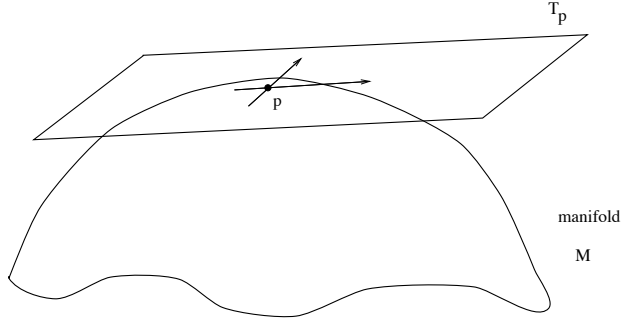


FIG. 1. Tangent space T_p of a two-dimensional manifold at point p . Reproduced from arXiv:gr-qc/9712019.

With this construction, the action of a dual vector on a vector is something we are quite familiar with

$$\begin{aligned} \omega(\mathbf{V}) &= \omega_\mu \mathbf{e}^{(\mu)}(V^\nu \mathbf{e}_{(\nu)}) \\ &= \omega_\mu V^\nu \mathbf{e}^{(\mu)}(\mathbf{e}_{(\nu)}) \\ &= \omega_\mu V^\nu \delta_\nu^\mu \\ &= \omega_\mu V^\mu, \end{aligned} \quad (3)$$

which is what we have been calling the inner (or dot) product between dual vector and a vector $\omega \cdot \mathbf{V}$. Note that this product is the same in every frame. A simple example of a dual vector is the gradient of a scalar function ϕ , which we will denote $d\phi$

$$d\phi = (\partial_\mu \phi) \mathbf{e}^{(\mu)}, \quad (4)$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. Upon a change of coordinates, the components of vectors and dual vectors transform as

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu, \quad \omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu, \quad (5)$$

respectively. The corresponding transformation laws for the coordinate basis vectors and dual vectors are

$$\mathbf{e}'_{(\mu)} = \frac{\partial x^\nu}{\partial x'^\mu} \mathbf{e}_{(\nu)}, \quad \mathbf{e}'^{(\mu)} = \frac{\partial x'^\mu}{\partial x^\nu} \mathbf{e}^{(\nu)}. \quad (6)$$

II. TENSORS

[What is a tensor?] Much like a dual vector is a linear map from vectors to $\mathbb{R}$, a tensor $\mathbf{T}$ of rank (k, l) is a multilinear map from k dual vectors and l vectors to the real numbers

$$\mathbf{T} : \underbrace{T_p^* \times \dots \times T_p^*}_{k \text{ times}} \times \underbrace{T_p \times \dots \times T_p}_{l \text{ times}} \rightarrow \mathbb{R}, \quad (7)$$

where $\times$ is the standard Cartesian product. You should think of the tensor $\mathbf{T}$ has a “machine” that “eats” k dual vectors and l vectors and returns a real number.

[Tensors act linearly on their arguments] As the “multilinear” characteristic implies, tensors act linearly on their arguments. For instance, for a rank $(1, 1)$ tensor $\mathbf{T}$, we have

$$\mathbf{T}(a\boldsymbol{\omega} + b\boldsymbol{\eta}, c\mathbf{V} + d\mathbf{W}) = ac\mathbf{T}(\boldsymbol{\omega}, \mathbf{V}) + ad\mathbf{T}(\boldsymbol{\omega}, \mathbf{W}) + bc\mathbf{T}(\boldsymbol{\eta}, \mathbf{V}) + bd\mathbf{T}(\boldsymbol{\eta}, \mathbf{W}) \in \mathbb{R}, \quad (8)$$

where a, b, c, d are real numbers, $\boldsymbol{\omega}, \boldsymbol{\eta}$ are dual vectors, and $\mathbf{V}, \mathbf{W}$ are vectors.

[Components of Tensors] Now, just like we can write vectors and dual vectors in terms of their components in a coordinate basis, we can write a rank (k, l) tensor in terms of its components in such a basis. To construct a basis for the space of all (k, l) tensors, we simply take the tensor product $\otimes$ of k coordinate basis vectors $\mathbf{e}_{(\mu)}$ and l coordinate basis dual vectors $\mathbf{e}^{(\nu)}$

$$\mathbf{e}_{(\mu_1)} \otimes \dots \otimes \mathbf{e}_{(\mu_k)} \otimes \mathbf{e}^{(\nu_1)} \otimes \dots \otimes \mathbf{e}^{(\nu_l)}. \quad (9)$$

With this basis, I can write my rank (k, l) tensor $\mathbf{T}$ in terms of its components as

$$\mathbf{T} = T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} \mathbf{e}_{(\mu_1)} \otimes \dots \otimes \mathbf{e}_{(\mu_k)} \otimes \mathbf{e}^{(\nu_1)} \otimes \dots \otimes \mathbf{e}^{(\nu_l)}. \quad (10)$$

[Transformation law for tensors] Now, the abstract tensor $\mathbf{T}$ must be the same in every frames. From Eq. (6) above, I know how my coordinate basis vectors and dual vectors transform under a change of coordinates. We thus have

$$\begin{aligned} \mathbf{T} &= T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} \mathbf{e}_{(\mu_1)} \otimes \dots \otimes \mathbf{e}_{(\mu_k)} \otimes \mathbf{e}^{(\nu_1)} \otimes \dots \otimes \mathbf{e}^{(\nu_l)} \\ &= T'^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} \mathbf{e}'_{(\mu_1)} \otimes \dots \otimes \mathbf{e}'_{(\mu_k)} \otimes \mathbf{e}'^{(\nu_1)} \otimes \dots \otimes \mathbf{e}'^{(\nu_l)} \\ &= T'^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \mathbf{e}_{(\alpha_1)} \otimes \dots \otimes \frac{\partial x^{\alpha_k}}{\partial x'^{\mu_k}} \mathbf{e}_{(\alpha_k)} \otimes \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \mathbf{e}^{(\beta_1)} \otimes \dots \otimes \frac{\partial x'^{\nu_l}}{\partial x^{\beta_l}} \mathbf{e}^{(\beta_l)}, \end{aligned} \quad (11)$$

from which we get

$$T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} = \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \dots \frac{\partial x^{\alpha_k}}{\partial x'^{\mu_k}} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \dots \frac{\partial x'^{\nu_l}}{\partial x^{\beta_l}} T'^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}. \quad (12)$$

Inverting this relationship, we get

$$T'^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\mu_k}}{\partial x^{\alpha_k}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\beta_l}}{\partial x'^{\nu_l}} T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l}. \quad (13)$$

Equation (13) is the *definition* of a rank (k, l) tensor. Any object that transforms like in Eq. (13) under a coordinate transformation $x \rightarrow x'$ will be a bona fide rank (k, l) tensor. Note: not every object having k upper indices and l lower indices is a tensor. We will see some examples of this later on.

[How tensors act on their arguments] Now that we know how component of tensors can be written down in coordinate bases, we can easily understand how a tensor $\mathbf{T}$ acts on its arguments. For instance, for a rank $(1, 2)$ tensor that takes in two vectors $\mathbf{V}, \mathbf{W}$ and one dual vectors $\boldsymbol{\omega}$ to return a real number, we have

$$\begin{aligned} \mathbf{T}(\boldsymbol{\omega}, \mathbf{V}, \mathbf{W}) &= T^{\mu}_{\nu\sigma} \mathbf{e}_{(\mu)} \otimes \mathbf{e}^{(\nu)} \otimes \mathbf{e}^{(\sigma)} (\omega_{\alpha} \mathbf{e}^{(\alpha)}, V^{\beta} \mathbf{e}_{(\beta)}, W^{\gamma} \mathbf{e}_{(\gamma)}) \\ &= T^{\mu}_{\nu\sigma} \mathbf{e}_{(\mu)} (\omega_{\alpha} \mathbf{e}^{(\alpha)}) \otimes \mathbf{e}^{(\nu)} (V^{\beta} \mathbf{e}_{(\beta)}) \otimes \mathbf{e}^{(\sigma)} (W^{\gamma} \mathbf{e}_{(\gamma)}) \\ &= T^{\mu}_{\nu\sigma} \omega_{\alpha} V^{\beta} W^{\gamma} \delta^{\alpha}_{\mu} \delta^{\nu}_{\beta} \delta^{\sigma}_{\gamma} \\ &= T^{\mu}_{\nu\sigma} \omega_{\mu} V^{\nu} W^{\sigma} \in \mathbb{R}, \end{aligned} \quad (14)$$

which is a rather intuitive way for tensors to act on vectors and dual vectors.

[No need to act on all their arguments] While we have defined tensors as multilinear maps from a set of vectors and dual vectors to the real numbers, we do not always have to act the tensor on its full set of arguments. For instance, for our rank $(1, 2)$ tensor $\mathbf{T}$ above, we can act it on a single vector $\mathbf{V}$ to form a perfectly fine rank $(1, 1)$ tensor $\mathbf{S}$, whose components are given by

$$S^\mu{}_\nu = T^\mu{}_{\nu\sigma} V^\sigma. \quad (15)$$

I could also act the tensor $\mathbf{T}$ on a rank $(2, 0)$ tensor $\mathbf{R}$ to form a perfectly fine vector $\mathbf{U}$

$$U^\mu = T^\mu{}_{\nu\sigma} R^{\nu\sigma}. \quad (16)$$

III. TENSORS OPERATIONS

A. Trace

The trace X of a $(1, 1)$ tensor $\mathbf{X}$ is

$$X = X^\nu{}_\nu. \quad (17)$$

If you think of $X^\nu{}_\mu$ has a matrix, then this is just the sum of the diagonal elements. However, if you are given $X_{\mu\nu}$, than X is not the sum of the diagonal elements of $X_{\mu\nu}$ since

$$X = X^\nu{}_\nu = g^{\nu\alpha} X_{\alpha\nu}. \quad (18)$$

A good example of this the trace of the Minkowski metric $\eta_{\mu\nu}$. We might naively think that its trace is $-1+1+1+1 = 2$, while in fact it is

$$\eta^{\mu\nu} \eta_{\mu\nu} = \delta^\mu{}_\mu = 4. \quad (19)$$

B. Contraction (or partial tracing)

We can contract one upper and one lower indices from a rank (k, l) tensor to form a $(k-1, l-1)$ tensor

$$S^{\mu\rho}{}_\sigma = T^{\mu\nu\rho}{}_{\sigma\nu}. \quad (20)$$

In general, which indices are contracted together is important; different choices will lead to different results, i.e.

$$T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu}. \quad (21)$$

C. Addition

You can add two rank (k, l) tensors to form another rank (k, l) tensor

$$T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = S^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + U^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}. \quad (22)$$

D. Product

We can create a rank $(k_1 + k_2, l_1 + l_2)$ tensor by taking the tensor product between a rank (k_1, l_1) tensor and a rank (k_2, l_2) tensor

$$T^{\mu_1 \dots \mu_{k_1+k_2}}{}_{\nu_1 \dots \nu_{l_1+l_2}} = S^{\mu_1 \dots \mu_{k_1}}{}_{\nu_1 \dots \nu_{l_1}} \otimes U^{\mu_{k_1+1} \dots \mu_{k_1+k_2}}{}_{\nu_{l_1+1} \dots \nu_{l_1+l_2}}. \quad (23)$$

E. Lowering or raising indices

We can use the metric $g_{\mu\nu}$ and inverse metric $g^{\mu\nu}$ to raise or lower indices, i.e.

$$T^{\mu\nu\rho}_{\sigma\epsilon} = g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g_{\sigma\delta} g_{\epsilon\phi} T_{\alpha\beta\gamma}^{\delta\phi}. \quad (24)$$

F. Symmetric and antisymmetric tensors

A tensor is said to be symmetric in two of its first and third indices if

$$S_{\mu\rho\nu} = S_{\nu\rho\mu}. \quad (25)$$

Similarly, a tensor is said to be symmetric in its two first indices if

$$S_{\mu\rho\nu} = S_{\rho\mu\nu}. \quad (26)$$

A tensor is to be symmetric if it is unchanged under all possible permutations of its indices.

$$S_{\mu\rho\nu} = S_{\mu\nu\rho} = S_{\rho\mu\nu} = S_{\rho\nu\mu} = S_{\nu\rho\mu} = S_{\nu\mu\rho}. \quad (27)$$

For instance, the metric is a symmetric $(0,2)$ tensor since $g_{\mu\nu} = g_{\nu\mu}$. A tensor is said to be antisymmetric in two of its first and third indices if

$$S_{\mu\rho\nu} = -S_{\nu\rho\mu}. \quad (28)$$

Similarly, a tensor is said to be antisymmetric in its two first indices if

$$S_{\mu\rho\nu} = -S_{\rho\mu\nu}. \quad (29)$$

If a tensor is said to be antisymmetric in all its indices, we just call this tensor antisymmetric (or completely antisymmetric).

G. Partial Derivatives

In general, the partial derivative of a tensor is **not** a tensor. For instance,

$$T_{\alpha}^{\mu}{}_{\nu} = \partial_{\alpha} R^{\mu}{}_{\nu} \quad (30)$$

is not a tensor in a general spacetime (however, it is in flat spacetime). We will soon see how to generalize the notion of derivative in curved spacetime.