

General Relativity Extra Problems 5

Question 1.

Part a)

Just from the properties of our $R_{\rho\sigma\mu\nu}$ tensor we can calculate that any component where $\rho = \sigma$ or $\mu = \nu$ must equal zero. To see this, take the example where $\rho = \sigma = \theta$, then:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

$$R_{\theta\theta\mu\nu} = -R_{\theta\theta\mu\nu}$$

$$R_{\theta\theta\mu\nu} = 0.$$

We end up with 12 components that are 0 in this way (see Tab. 1).

We have $R^\theta_{\phi\theta\phi} = \sin^2 \theta$. The first step is to lower the only upper index via:

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\alpha\theta} R^\alpha_{\phi\theta\phi} \\ &= g_{\theta\theta} R^\theta_{\phi\theta\phi} + g_{\phi\theta} R^\phi_{\phi\theta\phi} \\ &= a^2 \sin^2 \theta \end{aligned}$$

as $g_{\theta\theta} = a^2$ and $g_{\theta\phi} = 0$. From this we can use the properties of our tensor to calculate the remaining 3 components:

$$\begin{aligned} R_{\theta\phi\phi\theta} &= -R_{\theta\phi\theta\phi} \\ &= -a^2 \sin^2 \theta \end{aligned}$$

$$\begin{aligned} R_{\phi\theta\theta\phi} &= R_{\theta\phi\phi\theta} \\ &= -a^2 \sin^2 \theta \end{aligned}$$

$$\begin{aligned} R_{\phi\theta\phi\theta} &= -R_{\theta\phi\phi\theta} \\ &= a^2 \sin^2 \theta. \end{aligned}$$

The results are summarized in Tab. 1.

Index 1	Index 2	Index 4	Index 4	Value
θ	θ	θ	θ	0
θ	θ	θ	ϕ	0
θ	θ	ϕ	θ	0
θ	θ	ϕ	ϕ	0
θ	ϕ	θ	θ	0
θ	ϕ	θ	ϕ	$+a^2 \sin \theta$
θ	ϕ	ϕ	θ	$-a^2 \sin \theta$
θ	ϕ	ϕ	ϕ	0
ϕ	θ	θ	θ	0
ϕ	θ	θ	ϕ	$-a^2 \sin \theta$
ϕ	θ	ϕ	θ	$+a^2 \sin \theta$
ϕ	θ	ϕ	ϕ	0
ϕ	ϕ	θ	θ	0
ϕ	ϕ	θ	ϕ	0
ϕ	ϕ	ϕ	θ	0
ϕ	ϕ	ϕ	ϕ	0

TABLE 1. Values of each component of $R_{\rho\sigma\mu\nu}$.

Part b)

To start we need to calculate the inverse metric. Because our metric is diagonal each entry of the inverse metric is the reciprocal of the corresponding entry in the original metric. This gives us that:

$$\begin{aligned} g^{\theta\theta} &= \frac{1}{a^2} \\ g^{\phi\phi} &= \frac{1}{a^2 \sin^2 \theta} \\ g^{\theta\phi} &= g^{\phi\theta} = 0 \end{aligned}$$

Now we can calculate each component of $\tilde{R}_{\mu\nu}$ using the values for $R_{\rho\sigma\alpha\beta}$ we calculated in part a.

$$\begin{aligned} \tilde{R}_{\theta\theta} &= g^{\alpha\beta} R_{\beta\theta\alpha\theta} \\ &= g^{\theta\theta} R_{\theta\theta\theta\theta} + g^{\phi\phi} R_{\phi\theta\phi\theta} \\ &= \frac{1}{a^2}(0) + \frac{1}{a^2 \sin^2 \theta}(a^2 \sin^2 \theta) \\ &= 1 \\ \tilde{R}_{\theta\phi} &= g^{\alpha\beta} R_{\beta\theta\alpha\phi} \\ &= g^{\theta\theta} R_{\theta\theta\theta\phi} + g^{\phi\phi} R_{\phi\theta\phi\phi} \\ &= \frac{1}{a^2}(0) + \frac{1}{a^2 \sin^2 \theta}(0) \\ &= 0 \\ \tilde{R}_{\phi\theta} &= g^{\alpha\beta} R_{\beta\phi\alpha\theta} \\ &= g^{\theta\theta} R_{\theta\phi\theta\theta} + g^{\phi\phi} R_{\phi\phi\phi\theta} \\ &= \frac{1}{a^2}(0) + \frac{1}{a^2 \sin^2 \theta}(0) \\ &= 0 \\ \tilde{R}_{\phi\phi} &= g^{\alpha\beta} R_{\beta\phi\alpha\phi} \\ &= g^{\theta\theta} R_{\theta\phi\theta\phi} + g^{\phi\phi} R_{\phi\phi\phi\phi} \\ &= \frac{1}{a^2}(a^2 \sin^2 \theta) + \frac{1}{a^2 \sin^2 \theta}(0) \\ &= \sin^2 \theta \end{aligned}$$

Part c)

To calculate $R^\mu{}_\mu$ we need to raise the first index of the two diagonal components of $R_{\mu\nu}$:

$$\begin{aligned} R^\theta{}_\theta &= g^{\theta\alpha} R_{\alpha\theta} \\ &= \frac{1}{a^2}(1) + (0)(0) \\ &= \frac{1}{a^2} \\ R^\phi{}_\phi &= g^{\phi\alpha} R_{\alpha\phi} \\ &= (0)(0) + \frac{1}{a^2 \sin^2 \theta} \sin^2 \theta \\ &= \frac{1}{a^2}. \end{aligned}$$

$R^\mu{}_\mu$ is then $R^\theta{}_\theta + R^\phi{}_\phi = \frac{2}{a^2}$.