ST: General Relativity Extra Problem 8

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Q1. Consider the Poincaré half-plane, which has for metric

$$ds^2 = \frac{a^2}{y^2}\left(dx^2 + dy^2\right),$$

with $y > 0$, and where a is a constant.

(**a**). Compute the length of a $x =$constant line segment between the coordinates y_1 and y_2, with $y_2 > y_1$. Could an observer reach $y = 0$ by traveling a finite distance.

(**b**). Show that the geodesics in this space are either semi-circles with centers located on the $x-$axis or $x =$constant lines.

(**c**). Is this space curved? Is this a maximally symmetric space?.

Sol.

We are given that

$$g_{\mu\nu} = \begin{pmatrix} \frac{a^2}{y^2} & \\ & \frac{a^2}{y^2} \end{pmatrix} \quad \& \quad g^{\mu\nu} = \begin{pmatrix} \frac{y^2}{a^2} & \\ & \frac{y^2}{a^2} \end{pmatrix}.$$

We list all non-zero derivatives of metric tensor

$$\partial_y g_{xx} = -\frac{2a^2}{y^3}, \quad \partial_y g_{yy} = -\frac{2a^2}{y^3}.$$

Let's compute Christoffel symbols first

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right),$$

setting $\alpha = x$ one has

$$\Gamma^x_{\mu\nu} = \frac{1}{2} g^{xx} \left(\partial_\mu g_{\nu x} + \partial_\nu g_{x\mu} - \partial_x g_{\mu\nu} \right)$$

the only non-zero terms are $\mu = y,\ \nu = x$ or $\mu = x,\ \nu = y$

$$\Gamma^x_{xy} = \Gamma^x_{yx} = \frac{1}{2} g^{xx} \left(\partial_y g_{xx} \right) = -\frac{1}{y},$$

likewise, setting $\alpha = y$ we have

$$\Gamma^y_{\mu\nu} = \frac{1}{2} g^{yy} \left(\partial_\mu g_{\nu y} + \partial_\nu g_{y\mu} - \partial_y g_{\mu\nu} \right)$$

the only non-zero terms are $\mu = x,\ \nu = x$ and $\mu = y,\ \nu = y$

$$\Gamma^y_{xx} = \frac{1}{2} g^{yy} \left(-\partial_y g_{xx} \right) = \frac{1}{y} \quad \& \quad \Gamma^y_{yy} = \frac{1}{2} g^{yy} \left(\partial_y g_{yy} \right) = -\frac{1}{y}.$$

Geodesic equation is given by

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0,$$

Caveat! Since we are not in Minkowski space, thus we don't have such relation $d\tau = -ds^2$, thus instead using proper time $d\tau$ as variable for geodesic equation, it would be more appropriate to use ds as variable. Hence

$$\begin{cases} \dfrac{d^2 x}{ds^2} - \dfrac{2}{y} \dfrac{dx}{ds} \dfrac{dy}{ds} = 0 & (1) \\ \dfrac{d^2 y}{ds^2} + \dfrac{1}{y} \left(\left(\dfrac{dx}{ds} \right)^2 - \left(\dfrac{dy}{ds} \right)^2 \right) = 0 & (2) \end{cases}$$

let's focus on eq (1), one may realize that integral factor is y^{-2}, thus eq (1) read

$$\frac{d}{ds} \left(\frac{dx}{ds} y^{-2} \right) = 0 \implies \frac{dx}{ds} = C y^2, \ \ C \in \mathbb{R}.$$

(**a**). Now if $C = 0$, we get $x =$constant, in this case geodesic is simplify vertical line, thus

$$\ell = \int_{y_1}^{y_2} \sqrt{ds^2} = \int_{y_1}^{y_2} \frac{a}{y} \sqrt{\left(\frac{dx}{ds}\frac{ds}{dy}\right)^2 + 1}\, dy = a \int_{y_1}^{y_2} \frac{dy}{y} = a \log \frac{y_2}{y_1}$$

i.e., the length of $x =$constant line segment between y_1 and y_2 is given by $a \log \frac{y_2}{y_1}$, notice that observer would never reach $y = 0$ by traveling finite distance. □

(**b**). We have shown in part (**a**) that if $C = 0$ geodesics is just vertical lines, next assume $C \neq 0$, then consider following equations

$$\begin{cases} \dfrac{dx}{ds} = Cy^2 & (3) \\ \dfrac{a^2}{y^2}\left(\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2\right) = 1 & (4) \end{cases}$$

use the fact that $\frac{dy}{ds} = \frac{dy}{dx}\frac{dx}{ds}$, we may solve $\frac{dy}{dx}$ from equation (3) and (4),

$$\frac{dy}{dx} = \sqrt{\frac{y^2 - a^2C^2y^4}{a^2C^2y^4}} = \sqrt{\frac{1 - a^2C^2y^2}{a^2C^2y^2}}$$

$$\frac{aCy\, dy}{\sqrt{1 - a^2C^2y^2}} = dx$$

integrate on both sides yields

$$-\frac{\sqrt{1 - a^2C^2y^2}}{aC} + a = x, \quad a \in \mathbb{R}$$

$$\frac{1}{a^2C^2}\left(1 - a^2C^2y^2\right) = (x - a)^2$$

$$(x - a)^2 + y^2 = \frac{1}{a^2C^2}, \quad a, C \in \mathbb{R}\backslash\{0\}$$

as one may realize, this is semi-circles with centers located on the $x-$axis, as we see in following picture. □

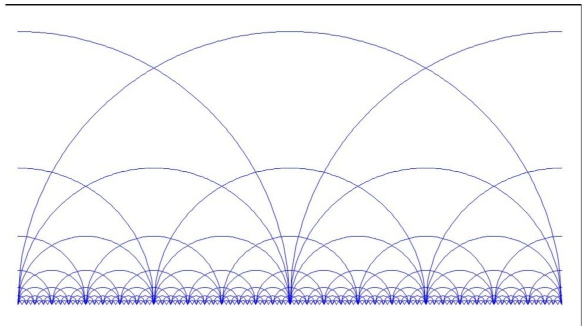

Figure 1: Geodesic in Poincaré Half-plane.

(**c**). Let's consider Riemann tensor, notice that the only non-zero term would be $xyxy$ (or $yxyx$),

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\mu\zeta}\Gamma^{\zeta}_{\beta\nu} - \Gamma^{\alpha}_{\nu\zeta}\Gamma^{\zeta}_{\beta\mu}$$

thus

$$\begin{aligned}
R^{x}{}_{yxy} &= \partial_x \Gamma^{x}_{yy} - \partial_y \Gamma^{x}_{yx} + \Gamma^{x}_{x\zeta}\Gamma^{\zeta}_{yy} - \Gamma^{x}_{y\zeta}\Gamma^{\zeta}_{yx} \\
&= -\partial_y \Gamma^{x}_{yx} + \Gamma^{x}_{xy}\Gamma^{y}_{yy} - \Gamma^{x}_{yx}\Gamma^{x}_{yx} \\
&= -\frac{1}{y^2} + \frac{1}{y^2} - \frac{1}{y^2} \\
&= -\frac{1}{y^2} \\
R^{y}{}_{xyx} &= g^{yy} g_{xx} R^{x}{}_{yxy} \\
&= -\frac{1}{y^2}
\end{aligned}$$

notice that $R^{x}{}_{yyx} = -R^{x}{}_{yxy}$ and $R^{y}{}_{xxy} = -R^{y}{}_{xyx}$, in fully covariant form, non-zero components are

$$R_{xyxy} = R_{yxyx} = -\frac{a^2}{y^4} \quad \& \quad R_{xyyx} = R_{yxxy} = \frac{a^2}{y^4}$$

Ricci tensor is given by

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

all that survives are two components

$$R_{xx} = R_{yy} = -\frac{1}{y^2},$$

finally Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} = -\frac{2}{a^2}.$$

The space has negative curvature, hence it is a *hyperbolic* like space. Finally, Maximally symmetric space is a space that is both homogeneous and isotropic, such space has largest number of Killing vector fields given by $\frac{N(N+1)}{2}$, where $N = \dim M$ is dimension of our manifold, further the following conditions would hold for such space

Ricci scalar R is a constant.

Ricci tensor is proportional to the metric tensor, $R_{\mu\nu} = \frac{R}{N} g_{\mu\nu}$.

Riemann curvature tensor is given by $R_{\alpha\beta\mu\nu} = \frac{R}{N(N-1)} \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \right).$

It is easy to see that Poincaré half-plane is maximally symmetric space. □