Extra Problem 10 Solution

April 11, 2024

1 Problem 1

See section III.C in the weak field-limit notes.

2 Problem 2

Starting from $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + A_{\mu\nu}\cos(\omega t - \vec{k}\cdot\vec{r})$, we have that

$$\begin{aligned}\Box^2 h_{\mu\nu} &= \partial^\rho \partial_\rho h_{\mu\nu} \\ &= g^{\sigma\rho}\partial_\rho\partial_\sigma h_{\mu\nu} \\ &= (\eta^{\sigma\rho} + h^{\sigma\rho})\partial_\rho\partial_\sigma h_{\mu\nu}\end{aligned}$$

where we can neglect $h^{\rho\sigma}$ because any terms containing it will be second order in $h_{\mu\nu}$. This gives us

$$\begin{aligned}\Box^2 h_{\mu\nu} &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{\mu\nu} \\ &= (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)A_{\mu\nu}\cos(\omega t - \vec{k}\cdot\vec{r}) \\ &= (-\omega^2 + |\vec{k}|^2)A_{\mu\nu}\cos(\omega t - \vec{k}\cdot\vec{r}).\end{aligned}$$

Where

$$\Box^2 h_{\mu\nu} = (-\omega^2 + |\vec{k}|^2)A_{\mu\nu}\cos(\omega t - \vec{k}\cdot\vec{r}) = 0$$

for non-trivial $h_{\mu\nu}$ only if $\omega^2 = |\vec{k}|^2$.

Now we impose

$$H_\nu = 0,$$

and we find

$$\begin{aligned}H_\nu &= \eta^{\alpha\mu}(\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial_n u h_{\alpha\mu}) \\ &= \eta^{\alpha\mu}(\partial_\mu A_{\alpha\nu}\cos(\omega t - \vec{k}\cdot\vec{r}) - \frac{1}{2}\partial_\nu A_{\alpha\mu}\cos(\omega t - \vec{k}\cdot\vec{r})) \\ &= \eta^{\alpha\mu}(-A_{\alpha\nu}B_\mu \sin(\omega t - \vec{k}\cdot\vec{r}) + \frac{1}{2}A_{\alpha\mu}B_\nu \sin(\omega t - \vec{k}\cdot\vec{r})),\end{aligned}$$

where $B_\nu = (\omega, -\vec{k})$. By now factoring out $-\sin(\omega t - \vec{k} \cdot \vec{r})$, we have

$$\eta^{\alpha\mu}(A_{\alpha\nu} B_\mu - \frac{1}{2} A_{\alpha\mu} B_\nu) = 0$$

$$A^\mu_\nu B_\mu = \frac{1}{2} A^\mu_\mu B_\nu.$$