PHYS 480/581
General Relativity

Extra Problems #11

Question 1.

Let us derive the Tolman-Oppenheimer-Volkoff equation, which is the General relativistic version of the hydrostatic equilibrium equation. To do so, we need to find a solution to the Einstein that is valid *inside* a static, spherically symmetry, non-rotating objects like a star. This is thus not a vacuum solution, and we need to solve for the full Einstein equation

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T). \tag{1}$$

Using the same arguments as for the Schwarzschild solution, our trial metric will admit the diagonal form

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

We have already shown in class that the source term for a perfect fluid admits the form

$$T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = (\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)g_{\mu\nu}, \tag{3}$$

where ρ is the energy density, p is the pressure, and u^μ is the 4-velocity of the fluid. The fluid forming the object is at rest, and thus the spatial components of the fluid 4-velocity vanish. Spherical symmetry implies that $\rho = \rho(r)$ and $p = p(r)$. From the normalization of the 4-velocity, we have $u_t = -\sqrt{A}$.

(a) Show that the Einstein equation implies

$$R_{tt} = 4\pi G(\rho + 3p)A, \tag{4}$$
$$R_{rr} = 4\pi G(\rho - p)B, \tag{5}$$
$$R_{\theta\theta} = 4\pi G(\rho - p)r^2, \tag{6}$$

and thus that

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} + 2\frac{R_{\theta\theta}}{r^2} = 16\pi G\rho. \tag{7}$$

(b) Use the Ricci tensor components for the above trial metric to show that the above equation reduces to

$$\frac{d}{dr}\left[r\left(1 - \frac{1}{B}\right)\right] = 8\pi G\rho r^2. \tag{8}$$

(c) Integrate both sides to get

$$B(r) = \left[1 - \frac{2Gm(r)}{r}\right]^{-1}, \tag{9}$$

where

$$m(r) \equiv \int_0^r 4\pi\rho r^2 dr. \tag{10}$$

Is $m(r)$ the mass enclosed within a radius r?

(d) To find $A(r)$, it is easiest to use energy conservation $\nabla_\nu T^{\mu\nu} = 0$. Show that the $\mu = r$ component of this equation implies

$$\frac{1}{A}\frac{dA}{dr} = -\frac{2}{\rho + p}\frac{dp}{dr}. \tag{11}$$

(e) Derive the Tolman-Oppenheimer-Volkoff equation by solving the above for dp/dr and using the $\theta\theta$ component of the Einstein equation to solve for A.

$$\frac{dp}{dr} = -\frac{\rho + p}{r^2}\left[\frac{4\pi G p r^3 + Gm(r)}{1 - 2Gm(r)/r}\right]. \tag{12}$$