Coffee Hour # 11

a) $R_{tt} = 8\pi G\left(T_{tt} - \frac{1}{2} g_{tt} T\right)$

$u_t = -\sqrt{A}$

$u_t^2 = A$

$$T_{tt} - \frac{1}{2} g_{tt} T = (\rho + P) u_t^2 + \frac{1}{2}(\rho - P) g_{tt}$$

$$= A(\rho + P) + (-A)\frac{1}{2}(\rho - P)$$

$$= \frac{1}{2}\rho + \frac{3}{2}P = +\frac{1}{2}(\rho + 3P)$$

$$R_{tt} = \frac{8\pi G}{2}(\rho + 3P)A = \underline{\underline{4\pi G(\rho + 3P)A}}$$

$$R_{rr} = 8\pi G\left(T_{rr} - \frac{1}{2} g_{rr} T\right)$$

$$T_{rr} - \frac{1}{2} g_{rr} T = (\rho + P) u_r^2 + \frac{1}{2}(\rho - P) g_{rr}$$

$u_r^2 = 0$

$u_\theta^2 = 0$

$g_{rr} = B$

$g_{\theta\theta} = r^2$

$$T_{rr} - \frac{1}{2} g_{rr} T = 0 + \frac{1}{2}(\rho - P) B$$

$$T_{\theta\theta} - \frac{1}{2} g_{\theta\theta} T = 0 + \frac{1}{2}(\rho - P) r^2$$

$$\left[R_{rr} = 4\pi G(\rho - P) B \qquad R_{\theta\theta} = 4\pi G(\rho - P) r^2 \right]$$

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} + 2\frac{R_{\theta\theta}}{r^2} = 4\pi G\left[\rho + 3P + \rho - P + 2\rho - 2P\right]$$

$$= 4\pi G(4\rho) = 16\pi G\rho$$

$$\therefore \quad \frac{R_{tt}}{A} + \frac{R_{rr}}{B} + 2\frac{R_{\theta\theta}}{r^2} = 16\pi G\rho$$

b) $$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} = \frac{1}{Br}\left(\frac{1}{A}\frac{\partial A}{\partial r} + \frac{1}{B}\frac{\partial B}{\partial r}\right)$$ [Moore 23.7]

and

$$\frac{2R_{\theta\theta}}{r^2} = -\frac{1}{ABr}\frac{\partial A}{\partial r} + \frac{1}{rB^2}\frac{\partial B}{\partial r} + \frac{2}{r^2} - \frac{2}{Br^2}$$ [Moore 23.6c]

Combining these we find

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} + \frac{2R_{\theta\theta}}{r^2} = \frac{2}{B^2 r}\frac{\partial B}{\partial r} + \frac{2}{r^2} - \frac{2}{Br^2} = 16\pi G\rho$$

$$\Rightarrow \frac{r}{B^2}\frac{\partial B}{\partial r} + 1 - \frac{1}{B} = 8\pi G\rho r^2$$

$$\Rightarrow \frac{d}{dr}\left[r\left(1 - \frac{1}{B}\right)\right] = 8\pi G\rho r^2 \checkmark$$

c) $$\int_0^r \frac{d}{dr}\left[r\left(1-\frac{1}{B}\right)\right] dr = 2G\int_0^r 4\pi\rho r^2\, dr = 2G\, m(r)$$

$$r\left(1 - \frac{1}{B}\right) = 2G\, m(r)$$

$$B(r) = \left[1 - \frac{2G\, m(r)}{r}\right]^{-1}$$

d) $\nabla_\nu T^{r\nu} = \partial_\nu T^{r\nu} + \Gamma^r_{\alpha\rho} T^{\rho\alpha} + \Gamma^\nu_{\nu\rho} T^{r\rho}$

$$= \partial_r T^{rr} + \Gamma^r_{\alpha\rho} T^{\rho\alpha} + \Gamma^\nu_{\nu r} T^{rr}$$

for $T^{rr} = g^{rr} T_{rr} = \frac{1}{B} P$

and similarly $T^{tt} = \frac{\rho}{A}$ and $T^{\theta\theta} = \frac{P}{r^2}$

and $T^{\phi\phi} = \frac{P}{r^2 \sin^2\theta}$

$$\Rightarrow \partial_r\left(\frac{P}{B}\right) + \Gamma^r_{tt} T^{tt} + \Gamma^r_{rr} T^{rr} + \Gamma^r_{\theta\theta} T^{\theta\theta} + \Gamma^r_{\phi\phi} T^{\phi\phi}$$

$$+ \Gamma^t_{tr} T^{rr} + \Gamma^r_{rr} T^{rr} + \Gamma^\theta_{\theta r} T^{rr} + \Gamma^\phi_{\phi r} T^{rr}$$

Using the Christoffel symbols from Appendix A in Moore

$$\Rightarrow \partial_r\left(\frac{P}{B}\right) + \frac{1}{2B}\frac{\partial A}{\partial r}\frac{\rho}{A} + \frac{1}{B}\frac{\partial B}{\partial r}\frac{P}{B} - \frac{1}{B}\frac{P}{r} - \cancel{\frac{P}{Br}} + \frac{P}{B}\left(\frac{1}{2A}\frac{\partial A}{\partial r} + \cancel{\frac{1}{r}} + \cancel{\frac{1}{r}}\right) = 0$$

$$= \partial_r\left(\frac{P}{B}\right) + \frac{\rho}{2AB}\frac{\partial A}{\partial r} + \frac{P}{B^2}\frac{\partial B}{\partial r} + \frac{P}{2AB}\frac{\partial A}{\partial r} = 0$$

$$= \frac{1}{B}\frac{\partial P}{\partial r} - \cancel{\frac{P}{B^2}\frac{\partial B}{\partial r}} + \frac{\rho}{2AB}\frac{\partial A}{\partial r} + \cancel{\frac{P}{B^2}\frac{\partial B}{\partial r}} + \frac{P}{2AB}\frac{\partial A}{\partial r} = 0$$

$$\frac{(P+\rho)}{2AB}\frac{\partial A}{\partial r} + \frac{1}{B}\frac{\partial P}{\partial r} = 0$$

$$\frac{1}{A}\frac{\partial A}{\partial r} = -\frac{2}{\rho + P}\frac{\partial P}{\partial r}$$

e) $R_{\theta\theta} = 8\pi G\left(T_{\theta\theta} - \frac{1}{2} g_{\theta\theta} T\right)$

$$= 4\pi G(\rho - p) r^2$$

$$R_{\theta\theta} = -\frac{r}{2AB}\frac{\partial A}{\partial r} + \frac{r}{2B^2}\frac{\partial B}{\partial r} + 1 - \frac{1}{B} \quad \text{[Moore 23.6c]}$$

$$-\frac{r}{2AB}\frac{\partial A}{\partial r} = 4\pi G(\rho - p) r^2 - \frac{r}{2B^2}\frac{\partial B}{\partial r} - 1 + \frac{1}{B}$$

$$\frac{1}{A}\frac{\partial A}{\partial r} = -2B\left[4\pi G(\rho - p)\right] r + \frac{1}{B}\frac{\partial B}{\partial r} + \frac{2B}{r} - \frac{2}{r}$$

$$= -8\pi G \rho r B + 8\pi G p r B + \frac{1}{B}\frac{\partial B}{\partial r} + \frac{2B}{r} - \frac{2}{r}$$

$$= \frac{2B}{r^2}\left[-4\pi G \rho r^3 + 4\pi G p r^3 + \frac{r^2}{2B^2}\frac{\partial B}{\partial r} + r - \frac{r}{B}\right]$$

$$= \frac{2B}{r^2}\left[-4\pi G \rho r^3 + 4\pi G p r^3 + r\left[\frac{r}{2B^2}\frac{\partial B}{\partial r} + 1 - \frac{1}{B}\right]\right]$$

$$= \frac{2B}{r^2}\left[-4\pi G \rho r^3 + 4\pi G p r^3 + \frac{r}{2}\left[\frac{d}{dr}\left[r\left(1 - \frac{1}{B}\right)\right]\right] - \frac{r^2}{2B^2}\frac{\partial B}{\partial r}\right]$$

for $\frac{d}{dr}\left[r\left(1 - \frac{1}{B}\right)\right] = 8\pi G \rho r^2$

$$\frac{2B}{r^2}\left[4\pi G p r^3 - \frac{r^2}{2B^2}\frac{\partial B}{\partial r}\right]$$

using $B = \left[1 - \frac{2Gm}{r}\right]^{-1}$

$$\frac{r^2}{2B^2}\frac{\partial}{\partial r}\left[1 - \frac{2Gm}{r}\right]^{-1} = \frac{r^2}{2B^2}\left[-\left[1 - \frac{2Gm}{r}\right]^{-2}\left(\frac{2Gm}{r^2}\right)\right]$$

$$= \frac{r^2}{2B^2}\left[-B^2\left(\frac{2Gm}{r^2}\right)\right] = -Gm$$

$$\therefore \frac{1}{A}\frac{\partial A}{\partial r} = \frac{2B}{r^2}\left[4\pi G \rho r^3 + Gm\right] = -\frac{2}{p+\rho}\frac{dp}{dr}$$

$$\therefore \frac{dp}{dr} = -\frac{p+\rho}{r^2} B\left[4\pi G \rho r^3 + Gm\right]$$

for $B = \left[1 - 2Gm/r\right]^{-1}$

$$\frac{dp}{dr} = -\frac{p+\rho}{r^2}\left[\frac{4\pi G \rho r^3 + Gm}{1 - 2Gm/r}\right]$$