# A curved and dark-energy dominated universe

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## 1

*Consider a curved universe with $\Omega_\Lambda > 1$ in which matter and radiation are negligible. Such a universe will never have a Big Bang singularity, but instead will have an instant of maximal (finite) density. Define that instant to be $t = 0$.*

### (a)

*Show that for such a universe, $a(t) = b cosh(\omega t)$, where $b = \sqrt{(\Omega_\Lambda - 1)/\Omega_\Lambda}$, and $\omega = H_0\sqrt{\Omega_\Lambda}$. If this universe is expanding at time $t_0$, will it ever cease expanding? If so, at what time?*

Start with the Friedmann equation (eq. 14 in the lecture notes) with only dark energy and curvature energy.

$$H^2 = H_0^2 \left( \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{\Omega_K}{a^2} \right) \tag{1}$$

$$= H_0^2 \left( \Omega_\Lambda + \frac{\Omega_K}{a^2} \right). \tag{2}$$

Setting this up was the physics. The rest is math.

#### Method 1: WolframAlpha or Mathematica

It is left as an exercise for the reader to integrate and invert

$$t(a) = \int^a \frac{\mathrm{d}a}{aH} = \frac{1}{H_0} \int^a \frac{\mathrm{d}a}{\sqrt{\Omega_\Lambda a^2 + \Omega_K}} \tag{3}$$

in WolframAlpha or Mathematica. (To derive the above, we technically have to use the fact that $H$ is positive, so $H = +\sqrt{H^2}$.)

#### Method 2: Take another time derivative

We get

$$2H\dot{H} = H_0^2 \Omega_K \left( \frac{-2\dot{a}}{a^3} \right) = H_0^2 \Omega_K \left( \frac{-2H}{a^2} \right) \tag{4}$$

$$\implies \dot{H} = -H_0^2 \Omega_K \frac{1}{a^2}. \tag{5}$$

Using

$$H_0^2 \Omega_K \frac{1}{a^2} = H^2 - H_0^2 \Omega_\Lambda \tag{6}$$

(from the Friedmann equation (2)), we get

$$\dot{H} = H_0^2 \Omega_\Lambda - H^2. \tag{7}$$

This is a nice intermediate form that can be integrated even more easily with WolframAlpha or Mathematica, but let's resist the temptation. Instead, plug in $H = \dot{a}/a$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_\Lambda - \left(\frac{\dot{a}}{a}\right)^2 \tag{8}$$

$$\implies \ddot{a} = (H_0^2\Omega_\Lambda)a. \tag{9}$$

This is a differential equation that physicists love. It has the general solution

$$a(t) = A\sinh(\omega t) + b\cosh(\omega t), \qquad \omega = H_0\sqrt{\Omega_\Lambda}. \tag{10}$$

(It gets sinh and cosh instead of sin and cos since the differential equation has a + sign.)

We have two degrees of freedom, which is one extra. This is because we lost information when we took another time derivative. We should plug our solution back into the original Friedmann equation to make sure it's good. First, though, we can take care of one degree of freedom with a trick. We know $a(t) > 0$, so our general solution will be a cosh form that we can shift along the $t$ axis. Since we've defined $t = 0$ to be at the minimum of the cosh (maximal density), we know what kind of shift we want. So we pick

$$a(t) = b\cosh(\omega t). \tag{11}$$

From here, the Friedman equation (2) says

$$\dot{a}^2 = H_0^2(\Omega_\Lambda a^2 + \Omega_K) \tag{12}$$

$$\implies \omega^2 b^2\sinh^2(\omega t) = H_0^2\Omega_\Lambda b^2\cosh^2(\omega t) + H_0^2\Omega_K \tag{13}$$

$$\implies 0 = H_0^2\Omega_\Lambda b^2 + H_0^2\Omega_K \tag{14}$$

$$\implies b = \sqrt{-\frac{\Omega_K}{\Omega_\Lambda}} = \sqrt{\frac{\Omega_\Lambda - 1}{\Omega_\Lambda}} \tag{15}$$

(using the definition of $\omega$, the property $\cosh^2(x) - \sinh^2(x) = 1$, and the fact $\Omega_\Lambda + \Omega_K = 1$).

Since $a(t) \propto \cosh(\omega t)$, once it is growing, it will grow forever.

## (b)

*Imagine that two observers in this universe determine from observations of their cosmic microwave background that $\Omega_\Lambda = 2$. How old is their universe at time $t_0$?*

Since $a = 1$ at the time they define as $t_0$,

$$1 = b\cosh(\omega t_0) \tag{16}$$

$$\implies t_0 = \frac{1}{\omega}\cosh^{-1}\left(\frac{1}{b}\right) \tag{17}$$

$$= \frac{1}{H_0\sqrt{\Omega_\Lambda}}\cosh^{-1}\sqrt{\frac{\Omega_\Lambda}{\Omega_\Lambda - 1}} \tag{18}$$

$$= \frac{1}{H_0}\frac{\cosh^{-1}\sqrt{2}}{\sqrt{2}} \tag{19}$$

$$\approx 0.623 \times \text{age of our universe}. \tag{20}$$

## (c)

*Is the spatial geometry of this universe spherical, flat, or saddle-like?*

Since $\Omega_K$ is defined to be $-\kappa/H_0^2$, negative $\Omega_K$ implies positive $\kappa$. Positive curvature $\kappa$ corresponds to a spherical, closed universe.

### (d)

*What is the curvature scale $R$ of this universe (which is the scale over which the spatial curvature of the universe becomes evident)? Remember that we define $\Omega_K = -\kappa/H_0^2$ and $\kappa = \pm 1/R^2$.*

$$R = \frac{1}{\sqrt{|\kappa|}} = \frac{1}{H_0}\frac{1}{\sqrt{|\Omega_K|}} = \frac{1}{H_0}\frac{1}{\sqrt{|\Omega_\Lambda - 1|}} \tag{21}$$

$$= \frac{1}{H_0} \qquad \text{(when } \Omega_\Lambda = 2\text{)}. \tag{22}$$

This is what the answer has to be (up to some numerical factors), since $H_0$ is the only important physical parameter with units in such a universe.