PHYS 480/581
General Relativity

Extra Problems #3

Question 1.

In the following $\Lambda^{\beta}{}_{\nu}$ is a Lorentz transformation, $\eta_{\mu\nu}$ is the Minkowski metric, and $\delta^{\mu}{}_{\nu}$ is the Kronecker delta.

(a) Show that $(\Lambda^{-1})^{\alpha}{}_{\mu}\eta_{\alpha\nu} = \eta_{\mu\beta}\Lambda^{\beta}{}_{\nu}$

(b) Compute $\delta^{\mu}{}_{\mu}$.

Question 2.

Here we consider the electromagnetic field strength tensor $F^{\mu\nu}$, which is antisymmetric in its two indices, that is, $F^{\mu\nu} = -F^{\nu\mu}$.

(a) Show that

$$\eta_{\mu\nu}F^{\mu\nu} = 0. \tag{1}$$

(b) In relativistic component notation, the electromagnetic Lorentz force law for a particle with electric charge q is

$$\frac{dp^{\mu}}{d\tau} = qF^{\mu\nu}\eta_{\nu\alpha}u^{\alpha}, \tag{2}$$

where p^{μ} is the four-momentum, τ is proper time, and u^{α} is the four-velocity. This equation implies that a charged particle moving in an electromagnetic field generally has a changing four-momentum (which matches our physical intuition). Show that this time-dependent four-momentum is nonetheless always consistent with the standard relation

$$\frac{d(p_{\mu}p^{\mu})}{d\tau} = 0, \tag{3}$$

which follows from the normalization of the four-momentum $p_{\mu}p^{\mu} = -m^2 = \text{constant}$, where m is the mass of the particle.