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PHYS 581
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General Relativity Extra Problems 3

Question 1.

Part a)

On question 3 of homework 2 we showed that:

$$\eta_{\sigma\nu} = \Lambda^{\mu}{}_{\sigma}\eta_{\mu\beta}\Lambda^{\beta}{}_{\nu}. \tag{1}$$

As $\delta^{\alpha}{}_{\sigma}\eta_{\alpha\nu} = \eta_{\sigma\nu}$, we can add a Kronecker delta to the left hand side of Eq. 1 and obtain:

$$\delta^{\alpha}{}_{\sigma}\eta_{\alpha\nu} = \Lambda^{\mu}{}_{\sigma}\eta_{\mu\beta}\Lambda^{\beta}{}_{\nu}. \tag{2}$$

Using the fact that:

$$\delta^{\alpha}{}_{\sigma} = (\Lambda^{-1})^{\alpha}{}_{\mu}\Lambda^{\mu}{}_{\sigma},$$

Eq. 2 becomes:

$$(\Lambda^{-1})^{\alpha}{}_{\mu}\Lambda^{\mu}{}_{\sigma}\eta_{\alpha\nu} = \Lambda^{\mu}{}_{\sigma}\eta_{\mu\beta}\Lambda^{\beta}{}_{\nu}. \tag{3}$$

Now all that remains is to divide both sides of Eq. 3 by $\Lambda^{\mu}{}_{\sigma}$ to obtain our result:

$$(\Lambda^{-1})^{\alpha}{}_{\mu}\eta_{\alpha\nu} = \eta_{\mu\beta}\Lambda^{\beta}{}_{\nu}. \tag{4}$$

□

Part b)

Remembering that:

$$\delta^{\mu}{}_{\nu} \equiv \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

we can show:

$$\begin{aligned} \delta^{\mu}{}_{\mu} &= \delta^{0}{}_{0} + \delta^{1}{}_{1} + \delta^{2}{}_{2} + \delta^{3}{}_{3} \\ &= 1 + 1 + 1 + 1 \\ &= 4. \end{aligned}$$

Question 2.

Part a)

We want to show that $\eta_{\mu\nu}F^{\mu\nu} = -\eta_{\mu\nu}F^{\mu\nu}$ and therefore $\eta_{\mu\nu}F^{\mu\nu} = 0$. Because $F^{\mu\nu}$ is antisymmetric ($F^{\mu\nu} = -F^{\nu\mu}$), we can write that:

$$\eta_{\mu\nu}F^{\mu\nu} = -\eta_{\mu\nu}F^{\nu\mu}. \tag{5}$$

Similarly, $\eta_{\mu\nu}$ is symmetric ($\eta_{\mu\nu} = \eta_{\nu\mu}$) and so:

$$\begin{aligned} \eta_{\mu\nu}F^{\mu\nu} &= -\eta_{\mu\nu}F^{\nu\mu} \\ &= -\eta_{\nu\mu}F^{\nu\mu}. \end{aligned} \tag{6}$$

Finally, as ν and μ are bound/dummy indices in this equation, we can flip their order without consequence (work this out if you don't believe it). This gives us:

$$\eta_{\mu\nu}F^{\mu\nu} = -\eta_{\mu\nu}F^{\mu\nu}, \tag{7}$$

which is only possible if $\eta_{\mu\nu}F^{\mu\nu} = 0$. □

Part b)

If we make the substitution $p^{\mu}p_{\mu} = \eta_{\mu\nu}p^{\nu}p^{\mu}$ our derivative becomes:

$$\frac{d}{d\tau}\left(p^{\mu}p_{\mu}\right) = \frac{d}{d\tau}\left(\eta_{\mu\nu}p^{\nu}p^{\mu}\right).$$

$\eta_{\mu\nu}$ has no τ dependence, and so applying the product rule yields:

$$\begin{aligned}\frac{d}{d\tau}\left(\eta_{\mu\nu}p^\nu p^\mu\right) &= \eta_{\mu\nu}\left(\frac{dp^\nu}{d\tau}p^\mu + p^\nu\frac{dp^\mu}{d\tau}\right)\\ &= \eta_{\mu\nu}\left(qF^{\nu\sigma}\eta_{\sigma\alpha}u^\alpha p^\mu + p^\nu qF^{\mu\sigma}\eta_{\sigma\alpha}u^\alpha\right).\end{aligned}$$

Now we take advantage of the fact that $p^\mu = mu^\mu$ before rearranging our terms:

$$\begin{aligned}\frac{d}{d\tau}\left(\eta_{\mu\nu}p^\nu p^\mu\right) &= \eta_{\mu\nu}\left(qF^{\nu\sigma}\eta_{\sigma\alpha}u^\alpha mu^\mu + mu^\nu qF^{\mu\sigma}\eta_{\sigma\alpha}u^\alpha\right)\\ &= mq\left(F^{\nu\sigma}\eta_{\sigma\alpha}\eta_{\mu\nu}u^\alpha u^\mu + F^{\mu\sigma}\eta_{\sigma\alpha}\eta_{\mu\nu}u^\nu u^\alpha\right)\\ &= mq\left(F^{\nu\sigma}u_\sigma u_\nu + F^{\mu\sigma}u_\mu u_\sigma\right)\\ &= mq\left(F^{\rho\sigma}u_\rho u_\sigma + F^{\rho\sigma}u_\rho u_\sigma\right).\end{aligned}$$

Now just as in part **a)** we use the facts that $F^{\rho\sigma} = -F^{\sigma\rho}$ and $F^{\rho\sigma}u_\rho u_\sigma = F^{\sigma\rho}u_\sigma u_\rho$ to write:

$$\begin{aligned}\frac{d}{d\tau}\left(\eta_{\mu\nu}p^\nu p^\mu\right) &= mq\left(F^{\rho\sigma}u_\rho u_\sigma - F^{\sigma\rho}u_\sigma u_\rho\right)\\ &= mq\left(F^{\rho\sigma}u_\rho u_\sigma - F^{\rho\sigma}u_\rho u_\sigma\right).\\ &= 0.\end{aligned}$$

□