The electromagnetic Lagrangian

a) $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

We want to show $\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu$

Plugging $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ into $\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}$,

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

Note that each term is only nonzero if $\partial_\alpha A_\beta = \partial_\mu A_\nu$, or $\partial_\beta A_\alpha = \partial_\mu A_\nu$, in which case it is 1.

$$= \underline{\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu}$$

b) We want to show $\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = 4F^{\mu\nu}$

Applying the product rule gives

$$\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}$$

We can pull out two metrics to lower the indices in the first term.

$$= F_{\alpha\beta}\eta^{\alpha\sigma}\eta^{\beta\lambda}\frac{\partial F_{\sigma\lambda}}{\partial(\partial_\mu A_\nu)} + F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}$$

Now we realize that if we use the metrics to raise the indices on $F_{\alpha\beta}$, both terms are equal.

$$= 2F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}$$

Plugging in $\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}$ from a).

$$= 2F^{\alpha\beta}(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu)$$

$$= 2(F^{\mu\nu} - F^{\nu\mu})$$

We know that $F^{\mu\nu} = -F^{\nu\mu}$.

$$= \underline{4F^{\mu\nu}}$$

c) We know $\dfrac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0$

We want to show $\partial_\mu F^{\mu\nu} = -J^\nu$

Let's start by plugging $\mathcal{L}$ into the above equation.

$$\frac{\partial}{\partial A_\nu}\Big(\underbrace{-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}}_{\text{This is independent of } A_\nu} + A_\alpha J^\alpha\Big) - \partial_\mu\left[\frac{\partial}{\partial(\partial_\mu A_\nu)}\Big(\underbrace{-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}}_{\text{From part b), we know this is } -F^{\mu\nu}} + \underbrace{A_\alpha J^\alpha}_{\text{This is independent of } \partial_\mu A_\nu}\Big)\right] = 0$$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial A_\nu}(A_\alpha J^\alpha)}_{=\,\delta^\nu_\alpha J^\alpha} + \partial_\mu F^{\mu\nu} = 0$$

$$\underline{\partial_\mu F^{\mu\nu} = -J^\nu}$$