PHYS 480/581
General Relativity

Extra Problems #5

Question 1.

The metric for a two-sphere is

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where a the the radius of the sphere. This metric tells you how to compute distances on the surface of a sphere of constant radius. Now, imagine that there is a rank-4 tensor $\boldsymbol{R}$ living on this two-sphere, with the following properties

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \qquad R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}, \tag{2}$$

and

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}. \tag{3}$$

where here $\rho, \sigma, \mu, \nu = \theta$ or ϕ. Say I give you one component of this tensor

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta. \tag{4}$$

(a) Using the properties given above and Eq. (4), find all 16 components of $\boldsymbol{R}$ with all four lower indices, i.e., $R_{\rho\sigma\mu\nu}$.

(b) Define the second-rank tensor $\tilde{\boldsymbol{R}}$ with components given by

$$\tilde{R}_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{5}$$

Find the four components of $\tilde{\boldsymbol{R}}$.

(c) Find the trace of $\tilde{\boldsymbol{R}}$, $\tilde{R}^{\mu}{}_{\mu}$.