PHYS 480/581
General Relativity

Extra Problems #6

Question 1.

Let's consider the metric

$$ds^2 = -dt^2 + [f(q)]^2 dq^2, \tag{1}$$

where $f(q)$ is an arbitrary function of the spatial coordinate q.

(a) Derive both the t and q components of the geodesic equation, using the proper time τ as the independent variable.

Solutions:

One way to do this is to first compute the Christoffel symbols. Since the metric is diagonal in the t, q coordinates, the Christoffels with an upper t index are

$$\Gamma^t_{\mu\nu} = \frac{1}{2} g^{tt} (\partial_\mu g_{\nu t} + \partial_\nu g_{t\mu} - \partial_t g_{\mu\nu}). \tag{2}$$

Given that $g_{tt} = -1$, $g_{tq} = 0$, and $\partial_t g_{qq} = 0$, this is always zero for all choices of μ, ν. Thus, $\Gamma^t_{tt} = \Gamma^t_{tq} = \Gamma^t_{qt} = \Gamma^t_{qq} = 0$. The Christoffels with an upper q index are

$$\Gamma^q_{\mu\nu} = \frac{1}{2} g^{qq} (\partial_\mu g_{\nu q} + \partial_\nu g_{q\mu} - \partial_q g_{\mu\nu}). \tag{3}$$

Thus,

$$\Gamma^q_{tt} = \frac{1}{2} g^{qq} (\partial_t g_{tq} + \partial_t g_{qt} - \partial_q g_{tt}) = 0. \tag{4}$$

$$\Gamma^q_{tq} = \frac{1}{2} g^{qq} (\partial_t g_{qq} + \partial_q g_{qt} - \partial_q g_{tq}) = \Gamma^q_{qt} = 0. \tag{5}$$

$$\begin{aligned}
\Gamma^q_{qq} &= \frac{1}{2} g^{qq} (\partial_q g_{qq} + \partial_q g_{qq} - \partial_q g_{qq}) \\
&= \frac{1}{2} g^{qq} \partial_q g_{qq} \\
&= \frac{1}{2 f^2(q)} 2 f(q) \frac{df(q)}{dq} \\
&= \frac{1}{f(q)} \frac{df(q)}{dq}
\end{aligned} \tag{6}$$

So, only one Christoffel connection coefficients is nonzero. So the two components of the geodesic equation are

$$\frac{d^2 t}{d\tau^2} = 0, \qquad \frac{d^2 q}{d\tau^2} + \frac{1}{f(q)} \frac{df(q)}{dq} \left(\frac{dq}{d\tau} \right)^2 = 0. \tag{7}$$

(b) Show that the t component of the geodesic equation implies that

$$\frac{dt}{d\tau} = \text{constant}, \tag{8}$$

Solutions:
The t component of the geodesic equation implies that

$$\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right) = 0 \Rightarrow \frac{dt}{d\tau} = \text{constant} = C. \tag{9}$$

(c) From the q component of the geodesic equation, show that

$$f\frac{dq}{d\tau} = \text{constant}. \tag{10}$$

Hint: use the fact that $\mathbf{u}\cdot\mathbf{u} = -1$, *with* $\mathbf{u} \equiv dx^\mu/d\tau$. Use the above to argue that the trajectory of a free particle in this spacetime obeys

$$\frac{dq}{dt} = \frac{\text{constant}}{f}. \tag{11}$$

Solutions:
For the q equation, we first need to realize that

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{qq}\left(\frac{dq}{d\tau}\right)^2 = -C^2 + f^2(q)\left(\frac{dq}{d\tau}\right)^2, \tag{12}$$

which implies that

$$f(q)\left(\frac{dq}{d\tau}\right) = \pm\sqrt{-1+C^2} = \text{constant}. \tag{13}$$

Thus,

$$\frac{dq}{dt} = \frac{dq}{d\tau}\frac{d\tau}{dt} = \pm\frac{\sqrt{-1+C^2}}{Cf(q)}. \tag{14}$$

(d) Define a new coordinate system (t, x) with $x = F(q)$, where F is the antiderivative of $f(q)$ (that is, $dF/dq = f(q)$). Show that the metric given in Eq. (1) above, once transformed to the (t, x) coordinates, is simply the metric for flat (2D) spacetime.
Solutions:
If we perform a coordinate transformation $x = F(q)$, such that $dF/dq = f(q)$. We then have

$$dx = \frac{dF}{dq}dq = f(q)dq \tag{15}$$

And the metric is then

$$ds^2 = -dt^2 + [f(q)]^2 dq^2 = -dt^2 + dx^2, \tag{16}$$

which is just the Minkowski metric. So, this spacetime is equivalent to flat spacetime.