ST: General Relativity Extra Problem 7

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Q1. Consider the metric of a three-sphere with coordinates $x^\mu = (\psi, \theta, \phi)$

$$ds^2 = d\psi^2 + \sin^2\psi \, \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{1}$$

(**a**). Calculate the Christoffel connection coefficients.

(**b**). Calculate the Riemann tensor components

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\mu}. \tag{2}$$

(**c**). Compute the Ricci tensor $R_{\mu\nu}$ and scalar R

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \quad R = R^\mu{}_\mu. \tag{3}$$

(**d**). Show that

$$R_{\alpha\beta\mu\nu} = \frac{R}{6}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right). \tag{4}$$

Spaces that satisfies this property are called *maximally symmetric*.

Sol.

We are given that

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & \sin^2\psi & \\ & & \sin^2\psi \, \sin^2\theta \end{pmatrix} \quad \& \quad g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & \sin^{-2}\psi & \\ & & \sin^{-2}\psi \, \sin^{-2}\theta \end{pmatrix}$$

we list all none zero derivative of metric tensor

$$\partial_\psi g_{\theta\theta} = 2\sin\psi\cos\psi, \quad \partial_\psi g_{\phi\phi} = 2\sin^2\theta\sin\psi\cos\psi, \quad \partial_\theta g_{\phi\phi} = 2\sin^2\psi\sin\theta\cos\theta.$$

(**a**). Recall Christoffel symbol is given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right),$$

by setting $\alpha = \psi$ and use the fact that metric tensor is diagonal, we see that the only non zero terms are

$$\Gamma^\psi_{\theta\theta} = \frac{1}{2}g^{\psi\psi}\left(-\partial_\psi g_{\theta\theta}\right) = -\sin\psi\cos\psi \quad \& \quad \Gamma^\psi_{\phi\phi} = \frac{1}{2}g^{\psi\psi}\left(-\partial_\psi g_{\phi\phi}\right) = -\sin^2\theta\sin\psi\cos\psi.$$

Likewise, let $\alpha = \theta$ the only non zero terms are

$$\Gamma^\theta_{\psi\theta} = \frac{1}{2}g^{\theta\theta}\left(\partial_\psi g_{\theta\theta}\right) = \sin^{-1}\psi\cos\psi \quad \& \quad \Gamma^\theta_{\phi\phi} = \frac{1}{2}g^{\theta\theta}\left(-\partial_\theta g_{\phi\phi}\right) = -\sin\theta\cos\theta.$$

Now, let $\alpha = \phi$ one has

$$\begin{aligned}
\Gamma^\phi_{\psi\phi} &= \frac{1}{2}g^{\phi\phi}\left(\partial_\psi g_{\phi\phi}\right) = \sin^{-2}\psi\ \sin^{-2}\theta\cdot\sin^2\theta\sin\psi\cos\psi = \sin^{-1}\psi\cos\psi \\
\Gamma^\phi_{\theta\phi} &= \frac{1}{2}g^{\phi\phi}\left(\partial_\theta g_{\phi\phi}\right) = \sin^{-2}\psi\ \sin^{-2}\theta\cdot\sin^2\psi\sin\theta\cos\theta = \sin^{-1}\theta\cos\theta.
\end{aligned}$$

Finally, according to symmetry property of Christoffel symbol we see all non zero Christoffel symbols are given by

$$\begin{cases}
\Gamma^\psi_{\theta\theta} = -\sin\psi\cos\psi \\
\Gamma^\psi_{\phi\phi} = -\sin^2\theta\sin\psi\cos\psi \\
\Gamma^\theta_{\theta\psi} = \Gamma^\theta_{\psi\theta} = \sin^{-1}\psi\cos\psi \\
\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \\
\Gamma^\phi_{\phi\psi} = \Gamma^\phi_{\psi\phi} = \sin^{-1}\psi\cos\psi \\
\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \sin^{-1}\theta\cos\theta.
\end{cases}$$

□

(**b**). We know that for n dimensional space we have $\frac{1}{12}n^2(n^2-1)$ independent components of Riemann tensor, back to our case $n=3$, thus we must have $\frac{3^2\cdot(3^2-1)}{12}=6$ independent components of Riemann tensor. We list all possible cases:

$$\psi\theta\psi\theta, \quad \psi\theta\psi\phi, \quad \psi\phi\psi\phi, \quad \psi\theta\phi\theta, \quad \psi\phi\theta\phi, \quad \theta\phi\theta\phi$$

now,

$$\begin{aligned}
R^{\psi}{}_{\theta\psi\theta} &= \partial_\psi \Gamma^\psi_{\theta\theta} - \partial_\theta \Gamma^\psi_{\theta\psi} + \Gamma^\psi_{\psi\gamma}\Gamma^\gamma_{\theta\theta} - \Gamma^\psi_{\theta\gamma}\Gamma^\gamma_{\theta\psi} \\
&= -\left(\cos^2\psi - \sin^2\psi\right) - \Gamma^\psi_{\theta\theta}\Gamma^\theta_{\theta\psi} \\
&= \sin^2\psi - \cos^2\psi + \sin\psi\cos\psi\cdot\sin^{-1}\psi\cos\psi \\
&= \sin^2\psi
\end{aligned}$$

likewise,

$$\begin{aligned}
R^{\psi}{}_{\theta\psi\phi} &= \partial_\psi \Gamma^\psi_{\theta\phi} - \partial_\phi \Gamma^\psi_{\theta\psi} + \Gamma^\psi_{\psi\gamma}\Gamma^\gamma_{\theta\phi} - \Gamma^\psi_{\phi\gamma}\Gamma^\gamma_{\theta\psi} \\
&= 0 \\
R^{\psi}{}_{\phi\psi\phi} &= \partial_\psi \Gamma^\psi_{\phi\phi} - \partial_\phi \Gamma^\psi_{\phi\psi} + \Gamma^\psi_{\psi\gamma}\Gamma^\gamma_{\phi\phi} - \Gamma^\psi_{\phi\gamma}\Gamma^\gamma_{\phi\psi} \\
&= \partial_\psi \Gamma^\psi_{\phi\phi} - \Gamma^\psi_{\phi\phi}\Gamma^\phi_{\phi\psi} \\
&= -\sin^2\theta\left(\cos^2\psi - \sin^2\psi\right) + \sin^2\theta\sin\psi\cos\psi\cdot\sin^{-1}\psi\cos\psi \\
&= \sin^2\theta\sin^2\psi \\
R^{\psi}{}_{\theta\phi\theta} &= \partial_\phi \Gamma^\psi_{\theta\theta} - \partial_\theta \Gamma^\psi_{\theta\phi} + \Gamma^\psi_{\phi\gamma}\Gamma^\gamma_{\theta\theta} - \Gamma^\psi_{\theta\gamma}\Gamma^\gamma_{\theta\phi} \\
&= 0 \\
R^{\psi}{}_{\phi\theta\phi} &= \partial_\theta \Gamma^\psi_{\phi\phi} - \partial_\phi \Gamma^\psi_{\phi\theta} + \Gamma^\psi_{\theta\gamma}\Gamma^\gamma_{\phi\phi} - \Gamma^\psi_{\phi\gamma}\Gamma^\gamma_{\phi\theta} \\
&= \partial_\theta \Gamma^\psi_{\phi\phi} + \Gamma^\psi_{\theta\theta}\Gamma^\theta_{\phi\phi} - \Gamma^\psi_{\phi\phi}\Gamma^\phi_{\phi\theta} \\
&= -2\sin\theta\cos\theta\sin\psi\cos\psi + \sin\psi\cos\psi\sin\theta\cos\theta + \sin^2\theta\sin\psi\cos\psi\cdot\sin^{-1}\theta\cos\theta \\
&= 0
\end{aligned}$$

and finally,

$$
\begin{aligned}
R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{\theta\gamma}\Gamma^{\gamma}_{\phi\phi} - \Gamma^{\theta}_{\phi\gamma}\Gamma^{\gamma}_{\phi\theta} \\
&= \partial_\theta \Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\theta\psi}\Gamma^{\psi}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} \\
&= -\left(\cos^2\theta - \sin^2\theta\right) - \sin^{-1}\psi\cos\psi\cdot\sin^2\theta\sin\psi\cos\psi + \sin\theta\cos\theta\cdot\sin^{-1}\theta\cos\theta \\
&= \sin^2\theta\left(1-\cos^2\psi\right) \\
&= \sin^2\theta\sin^2\psi.
\end{aligned}
$$

□

(**c**). We know that $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, thus

$$
\begin{aligned}
R_{\psi\psi} &= R^{\alpha}{}_{\psi\alpha\psi} = R^{\theta}{}_{\psi\theta\psi} + R^{\phi}{}_{\psi\phi\psi} \\
&= g^{\theta\theta} g_{\psi\psi} R^{\psi}{}_{\theta\psi\theta} + g^{\phi\phi} g_{\psi\psi} R^{\psi}{}_{\phi\psi\phi} \\
&= \sin^{-2}\psi\sin^2\psi + \sin^{-2}\psi\sin^{-2}\theta\sin^2\theta\sin^2\psi \\
&= 2 \\
R_{\theta\theta} &= R^{\alpha}{}_{\theta\alpha\theta} = R^{\psi}{}_{\theta\psi\theta} + R^{\phi}{}_{\theta\phi\theta} \\
&= R^{\psi}{}_{\theta\psi\theta} + g^{\phi\phi} g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\
&= \sin^2\psi + \sin^{-2}\psi\sin^{-2}\theta\sin^2\psi\sin^2\theta\sin^2\psi \\
&= 2\sin^2\psi \\
R_{\phi\phi} &= R^{\alpha}{}_{\phi\alpha\phi} = R^{\psi}{}_{\phi\psi\phi} + R^{\theta}{}_{\phi\theta\phi} \\
&= R^{\psi}{}_{\phi\psi\phi} + g^{\phi\phi} g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\
&= \sin^2\theta\sin^2\psi + \sin^2\theta\sin^2\psi \\
&= 2\sin^2\theta\sin^2\psi
\end{aligned}
$$

now consider non-diagonal elements

$$
\begin{aligned}
g_{\psi\theta} = R^{\alpha}{}_{\psi\alpha\theta} = R^{\phi}{}_{\psi\phi\theta} = g^{\phi\phi} g_{\psi\psi} R^{\psi}{}_{\phi\theta\phi} = 0, \\
g_{\psi\phi} = R^{\alpha}{}_{\psi\alpha\phi} = R^{\theta}{}_{\psi\theta\phi} = g^{\theta\theta} g_{\psi\psi} R^{\psi}{}_{\theta\phi\theta} = 0, \\
g_{\theta\phi} = R^{\alpha}{}_{\theta\alpha\phi} = R^{\psi}{}_{\theta\psi\phi} = 0.
\end{aligned}
$$

Finally, Ricci scalar is given by

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} = g^{\psi\psi} R_{\psi\psi} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} \\
&= 2 + 2\sin^{-2}\psi \sin^2\psi + 2\sin^{-2}\theta \sin^{-2}\psi \sin^2\theta \sin^2\psi \\
&= 6.
\end{aligned}
$$

□

(**d**). Actually part (d) follows from the consequence of **Killing vector fields**. Let's start with basic property of Killing vector field $\xi^{\nu}\mathbf{e}_{\nu}$,

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0, \quad \text{where} \quad \xi_{\mu} = g_{\mu\sigma}\xi^{\sigma}, \tag{5}$$

$$\nabla_{\rho}\nabla_{\nu}\xi_{\mu} - \nabla_{\nu}\nabla_{\rho}\xi_{\mu} = R^{\lambda}{}_{\mu\nu\rho}\xi_{\lambda} \tag{6}$$

$$\nabla_{\mu}\nabla_{\nu}\xi_{\rho} = R^{\lambda}{}_{\mu\nu\rho}\xi_{\lambda} \tag{7}$$

$$\nabla_{\nu}\xi_{\mu} = \xi_{\mu}\nabla_{\nu} \tag{8}$$

the equation (5) is so called **Killing** equation, $R^{\lambda}{}_{\mu\nu\rho}$ in equation (6) and (7) is just Riamann curvature tensor, and equation (7) basically tells us that Killing vector commute with covariant deraivative.

Now, consider Lie bracket $[\nabla_{\kappa}, \nabla_{\alpha}]$ acting on $\nabla_{\nu}\xi_{\mu}$, from equation (7) and product rule one has

$$
\begin{aligned}
[\nabla_{\kappa}, \nabla_{\alpha}]\nabla_{\nu}\xi_{\mu} &= \nabla_{\kappa}\nabla_{\alpha}\nabla_{\nu}\xi_{\mu} - \nabla_{\alpha}\nabla_{\kappa}\nabla_{\nu}\xi_{\mu} = \nabla_{\kappa}\left(R^{\beta}{}_{\alpha\nu\mu}\xi_{\beta}\right) - \nabla_{\alpha}\left(R^{\beta}{}_{\kappa\nu\mu}\xi_{\beta}\right) \\
&= \left(\nabla_{\kappa}R^{\beta}{}_{\alpha\nu\mu}\right)\xi_{\beta} + R^{\beta}{}_{\alpha\nu\mu}\nabla_{\kappa}\xi_{\beta} - \left(\nabla_{\alpha}R^{\beta}{}_{\kappa\nu\mu}\right)\xi_{\beta} - R^{\beta}{}_{\kappa\nu\mu}\nabla_{\alpha}\xi_{\beta} \\
&= \nabla_{\kappa}R^{\beta}{}_{\alpha\nu\mu}\xi_{\beta} + R^{\zeta}{}_{\alpha\nu\mu}\nabla_{\kappa}\xi_{\zeta} - \nabla_{\alpha}R^{\beta}{}_{\kappa\nu\mu}\xi_{\beta} - R^{\zeta}{}_{\kappa\nu\mu}\nabla_{\alpha}\xi_{\zeta}
\end{aligned} \tag{9}
$$

were last equation we just rename index $\beta \to \zeta$.

On the other hand by equation (6) and (8) we also have

$$\begin{aligned}
[\nabla_\kappa, \nabla_\alpha]\nabla_\nu \xi_\mu &= \nabla_\kappa \nabla_\alpha \nabla_\nu \xi_\mu - \nabla_\alpha \nabla_\kappa \nabla_\nu \xi_\mu = \nabla_\kappa \nabla_\alpha \left(\nabla_\nu \xi_\mu\right) - \nabla_\alpha \nabla_\kappa \left(\nabla_\nu \xi_\mu\right) \\
&= \nabla_\kappa \nabla_\alpha \left(\nabla_\nu\right) \xi_\mu + \nabla_\kappa \nabla_\alpha \left(\xi_\mu\right) \nabla_\nu - \nabla_\alpha \nabla_\kappa \left(\nabla_\nu\right) \xi_\mu - \nabla_\alpha \nabla_\kappa \left(\xi_\mu\right) \nabla_\nu \\
&= \nabla_\kappa \nabla_\alpha \left(\nabla_\nu\right) \xi_\mu - \nabla_\alpha \nabla_\kappa \left(\nabla_\nu\right) \xi_\mu + \nabla_\kappa \nabla_\alpha \left(\xi_\mu\right) \nabla_\nu - \nabla_\alpha \nabla_\kappa \left(\xi_\mu\right) \nabla_\nu \\
&= R^\beta{}_{\nu\alpha\kappa} \nabla_\beta \xi_\mu + R^\beta{}_{\mu\alpha\kappa} \xi_\beta \nabla_\nu \\
&= R^\beta{}_{\nu\alpha\kappa} \nabla_\beta \xi_\mu + R^\beta{}_{\mu\alpha\kappa} \nabla_\nu \xi_\beta \qquad (10)
\end{aligned}$$

thus, we must have equation (9) and (10) are identical, regrouping them yields

$$\left(\nabla_\kappa R^\beta{}_{\alpha\nu\mu} - \nabla_\alpha R^\beta{}_{\kappa\nu\mu}\right) \xi_\beta + R^\zeta{}_{\alpha\nu\mu} \nabla_\kappa \xi_\zeta - R^\zeta{}_{\kappa\nu\mu} \nabla_\alpha \xi_\zeta - R^\beta{}_{\mu\alpha\kappa} \nabla_\nu \xi_\beta - R^\beta{}_{\nu\alpha\kappa} \nabla_\beta \xi_\mu = 0$$

now, we use equation (5) changing second last term $-R^\beta{}_{\mu\alpha\kappa} \nabla_\nu \xi_\beta$ to $R^\beta{}_{\mu\alpha\kappa} \nabla_\beta \xi_\nu$:

$$\left(\nabla_\kappa R^\beta{}_{\alpha\nu\mu} - \nabla_\alpha R^\beta{}_{\kappa\nu\mu}\right) \xi_\beta + R^\zeta{}_{\alpha\nu\mu} \nabla_\kappa \xi_\zeta - R^\zeta{}_{\kappa\nu\mu} \nabla_\alpha \xi_\zeta + R^\beta{}_{\mu\alpha\kappa} \nabla_\beta \xi_\nu - R^\beta{}_{\nu\alpha\kappa} \nabla_\beta \xi_\mu = 0$$

we may also rename indices by multiplying kronecker delta,

$$\begin{aligned}
\left(\nabla_\kappa R^\beta{}_{\alpha\nu\mu} - \nabla_\alpha R^\beta{}_{\kappa\nu\mu}\right) \xi_\beta + R^\zeta{}_{\alpha\nu\mu} \delta^\beta_\kappa \nabla_\beta \xi_\zeta - R^\zeta{}_{\kappa\nu\mu} \delta^\beta_\alpha \nabla_\beta \xi_\zeta + R^\beta{}_{\mu\alpha\kappa} \delta^\zeta_\nu \nabla_\beta \xi_\zeta - R^\beta{}_{\nu\alpha\kappa} \delta^\zeta_\mu \nabla_\beta \xi_\zeta &= 0 \\
\left(\nabla_\kappa R^\beta{}_{\alpha\nu\mu} - \nabla_\alpha R^\beta{}_{\kappa\nu\mu}\right) \xi_\beta + \left(R^\zeta{}_{\alpha\nu\mu} \delta^\beta_\kappa - R^\zeta{}_{\kappa\nu\mu} \delta^\beta_\alpha + R^\beta{}_{\mu\alpha\kappa} \delta^\zeta_\nu - R^\beta{}_{\nu\alpha\kappa} \delta^\zeta_\mu\right) \nabla_\beta \xi_\zeta &= 0 \qquad (11)
\end{aligned}$$

we may further assume that $\xi_\beta \neq 0$ and $\nabla_\beta \xi_\zeta \neq 0$ for all possible β and ζ, so that equation (11) further implies that

$$\begin{cases}
\nabla_\kappa R^\beta{}_{\alpha\nu\mu} - \nabla_\alpha R^\beta{}_{\kappa\nu\mu} = 0 \\
R^\zeta{}_{\alpha\nu\mu} \delta^\beta_\kappa - R^\zeta{}_{\kappa\nu\mu} \delta^\beta_\alpha + R^\beta{}_{\mu\alpha\kappa} \delta^\zeta_\nu - R^\beta{}_{\nu\alpha\kappa} \delta^\zeta_\mu = 0
\end{cases}$$

let's focus on the second equation, by multiplying δ^{κ}_{β} one has

$$
\begin{aligned}
R^{\zeta}{}_{\alpha\nu\mu}\delta^{\beta}_{\kappa}\delta^{\kappa}_{\beta} - R^{\zeta}{}_{\kappa\nu\mu}\delta^{\beta}_{\alpha}\delta^{\kappa}_{\beta} + R^{\beta}{}_{\mu\alpha\kappa}\delta^{\zeta}_{\nu}\delta^{\kappa}_{\beta} - R^{\beta}{}_{\nu\alpha\kappa}\delta^{\zeta}_{\mu}\delta^{\kappa}_{\beta} &= 0 \\
R^{\zeta}{}_{\alpha\nu\mu}N - R^{\zeta}{}_{\kappa\nu\mu}\delta^{\kappa}_{\alpha} + R^{\beta}{}_{\mu\alpha\beta}\delta^{\zeta}_{\nu} - R^{\beta}{}_{\nu\alpha\beta}\delta^{\zeta}_{\mu} &= 0 \\
R^{\zeta}{}_{\alpha\nu\mu}N - R^{\zeta}{}_{\kappa\nu\mu}\delta^{\kappa}_{\alpha} - R^{\beta}{}_{\mu\beta\alpha}\delta^{\zeta}_{\nu} + R^{\beta}{}_{\nu\beta\alpha}\delta^{\zeta}_{\mu} &= 0 \\
R^{\zeta}{}_{\alpha\nu\mu}N - R^{\zeta}{}_{\alpha\nu\mu} - R_{\mu\alpha}\delta^{\zeta}_{\nu} + R_{\nu\alpha}\delta^{\zeta}_{\mu} &= 0 \\
(N-1)\, R^{\zeta}{}_{\alpha\nu\mu} &= -\delta^{\zeta}_{\mu}R_{\nu\alpha} + \delta^{\zeta}_{\nu}R_{\mu\alpha} \qquad (12)
\end{aligned}
$$

where $N = \delta^{\beta}_{\kappa}\delta^{\kappa}_{\beta} = \delta^{\kappa}_{\kappa}$ is the dimensional of our manifold (in this case dim $M = 3$). Next, multiplying metric tensor $g_{\beta\zeta}$ on both sides of (12),

$$
\begin{aligned}
(N-1)\, g_{\beta\zeta}R^{\zeta}{}_{\alpha\nu\mu} &= -g_{\beta\zeta}\delta^{\zeta}_{\mu}R_{\nu\alpha} + g_{\beta\zeta}\delta^{\zeta}_{\nu}R_{\mu\alpha} \\
(N-1)\, R_{\beta\alpha\nu\mu} &= -g_{\beta\mu}R_{\nu\alpha} + g_{\beta\nu}R_{\mu\alpha} \\
(N-1)\, R_{\alpha\beta\mu\nu} &= -g_{\beta\mu}R_{\nu\alpha} + g_{\beta\nu}R_{\mu\alpha} \qquad (13)
\end{aligned}
$$

we are almost there. The last few steps is first we multiply inverse metric tensor $g^{\alpha\mu}$ on both sides of (13) yields

$$
\begin{aligned}
(N-1)\, g^{\alpha\mu}R_{\alpha\beta\mu\nu} &= -\delta^{\alpha}_{\beta}R_{\nu\alpha} + g_{\beta\nu}\left(g^{\alpha\mu}R_{\mu\alpha}\right) \\
(N-1)\, R_{\beta\nu} &= -\delta^{\alpha}_{\beta}R_{\nu\alpha} + g_{\beta\nu}R \\
(N-1)\, R_{\beta\nu} &= -R_{\nu\beta} + g_{\beta\nu}R \\
(N-1)\, R_{\beta\nu} &= -R_{\beta\nu} + g_{\beta\nu}R \\
NR_{\beta\nu} - g_{\beta\nu}R &= -R_{\beta\nu} + R_{\beta\nu} \\
NR_{\beta\nu} - g_{\beta\nu}R &= -g_{\beta\nu}R + g_{\beta\nu}R \\
NR_{\beta\nu} &= g_{\beta\nu}R \\
\implies R_{\beta\nu} &= \frac{R}{N}g_{\beta\nu} \qquad (14)
\end{aligned}
$$

now plug equation (14) back into (13) we see that

$$\begin{aligned}
(N-1)\, R_{\alpha\beta\mu\nu} &= -g_{\beta\mu}\frac{R}{N}g_{\nu\alpha} + g_{\beta\nu}\frac{R}{N}g_{\mu\alpha} \\
(N-1)\, R_{\alpha\beta\mu\nu} &= \frac{R}{N}\left(g_{\beta\nu}g_{\mu\alpha} - g_{\beta\mu}g_{\nu\alpha}\right) \\
R_{\alpha\beta\mu\nu} &= \frac{R}{N(N-1)}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right)
\end{aligned}$$

finally we arrive the place that Riemann curvature tensor describes a maximally symmetric space, and part (d) follows immediately since in our case $N = 3$ hence

$$R_{\alpha\beta\mu\nu} = \frac{R}{6}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right).$$

□